Jonathan R. PARTINGTON, *An Introduction to Hankel Operators*, London Mathematical Society Student Texts 13, Cambridge University Press, Cambridge, 1988, v + 101 pages, Hard Cover, 20 PS (34.5 USD), Paperback 7.5 PS (12.95 USD).

Hankel matrices have always played an important role in linear system theory. A new impetus to the interest in Hankel matrices and operators was given at the end of the seventies when engineers realized that the paper of Adamjan, Arov and Krein on the *Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem*, which was published ten years earlier in 1971 in USSR Math. Sbornik. could be used to solve a problem of model reduction.

This method of approximating a transfer function of a system has since then be known as the AAK approximation, in honour of these Russian mathematicians. The idea is to associate with the given transfer function some Hankel operator. This Hankel operator, whose rank is the order of the system, is then approximated (in l_2 operator-norm) by a Hankel operator of lower rank. From this low rank Hankel operator, one can go back to a transfer function of a low order system which approximates the original one. Therefore the method is also known as Hankel-norm approximation. The result obtained by AAK is quite remarkable. It is known that by truncating the singular value decomposition (in the Russian literature, singular values are known as s-numbers and the left-right singular vectors are Schmidt pairs) after k terms, a best frank k approximation is obtained for the operator, but it is not at all trivial to see that this best approximant can be chosen to be a Hankel operator if the given operator is also Hankel. This is of course essential because the classical Kronecker theorem states that a Hankel matrix has finite rank iff its symbol (i.e. the function analytic in $|z| > 1$ with Fourier coefficients equal the Hankel matrix entries) is a rational function with the number of poles inside $|z| < 1$ equal to the rank of the matrix and Nehari's theorem says that the l_2-norm of the Hankel operator equals the l_∞-norm of its symbol. The latter implies that Hankel-norm approximations can be used to find near best l_∞ rational approximants. This problem was already studied by Carathéodory and Féjer. Because of this relation, also people from approximation theory became interested. They called it CF-approximation (Carathéodory-Féjer). In the last decennium there came an outburst of papers on the theory and the applications of these Hankel operators. There is a paper by K. Glover (1984) reformulating the system application in a state-space setting ; S.C. Power has a paper on Hankel operators in Hilbert spaces (1980) and a book on this topic (1982) ; V. Peller has a long paper on Hankel operators of class C_p (1982) ; J.W. Helton (1987) wrote some lecture notes on Operator theory, analytic functions, matrices and electrical engineering ; S.C. Power also edited in 1984 conference proceedings on the topic, etc. Most of these publications including the AAK paper itself give advanced results and require a sound mathematical basis from the reader. Since many of the potential users of the theory are engineers, a relatively simple but rigorous introduction to the theory is needed. With these objectives, the current lecture notes were written. This book is based on graduate lectures given to

mathematicians and control engineers. To make the book selfcontained, the author has included several appendices on mathematical topics unlikely to be met by undergraduate engineers. All this is kept very concise. On only 13 pages on which Banach spaces, operators, spectral theory, Stone-Weierstrass, Hahn-Banach theorems, measure theory, Fatou's lemma, Fubini's theorem, Hölder's equality, Jensen's unequality and residues are listed. The reader is expected to know complex and functional analysis.

The main body of this booklet starts after a short introduction with 7 sections of about 10 pages each. The first section treats compact operators on a Hilbert space. More precisely the singular value decomposition (Schmidt expansion) and its properties are given. A second section on Hardy spaces introduces the spaces L_p and H_p for the unit circle, Blaschke products, the Riesz factorization theorem and the Fourier transformation. Next Hardy spaces for the half plane are given with now the Laplace transform as an analogue of the Fourier transform. Section 3 finally introduces the basic properties of Hankel operators. The big theorems of Nehari and Kronecker are introduced here. Also two famous (extension, interpolation) problems from complex analysis : Carathéodory-Féjer and Nevanlinna-Pick are formulated in their simplest form. It is shown that both can be reduced to the Nehari extension problem, i.e. finding a symbol of minimum norm. Sarason's elegant solution for this problem is proposed. Also Hartman's theorem characterizing compact Hankel operators is presented. All this section is presented in the unit circle setting. Next section treats Hankel operators on the half plane, i.e. the Hankel integral operator which is the continuous analog of the (discrete) infinite Hankel matrix. Several results from the previous section are transcribed in the new setting. This section is therefore very short (6 pages). The next 7 pages contain section 5, introducing linear systems and how they tie up with H_∞. The sixth section introduces the Hankel norm approximation problem, that is the celebrated AAK results. This section could be the end point of the book : Kronecker's theorem, Blaschke products and Beurling's theorem on closed shift-invariant subspaces of H_2 leads to the result. There is yet another section on special classes of Hankel operators. The singular values of Hankel operators which play such an important role in approximation problems are investigated in a more general context. Singular values, nuclear and Hilbert-Schmidt Hankel operators are discussed in more detail. The emphasis is on the (continuous) Hankel integral operator. Most of the results given here are found in a paper by Bonsall and Walsh or Glover and Partington. The booklet ends with 45 exercises, a short, but carefully chosen set of books and articles with a brief comment and a subject index.

This book brings together some results from the recent literature on this topic in a simple treatment that will make them accessible to a wide audience, including graduate students in mathematics or engineering.

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