Model order reduction for PDE constrained optimization in vibrations

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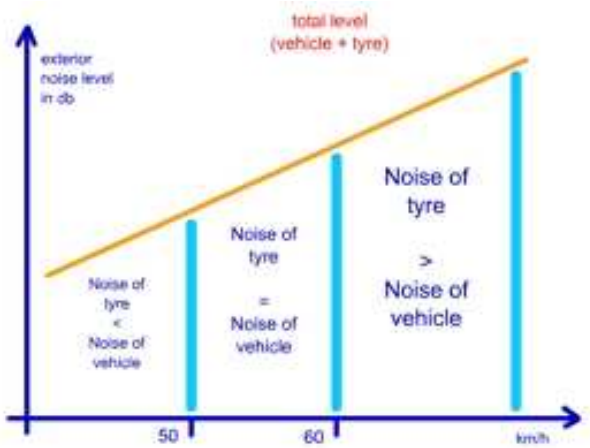
Examples of vibrating systems

- Car tyres





- Windscreens



  - Structural damping
  - Choice of connection (glue) to the car

Examples of vibrating systems

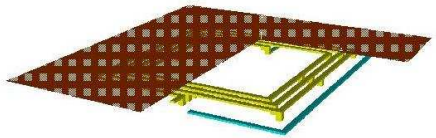
- Planes



- Bridge vibrating under footsteps and Thames wind



- Maxwell-equation – electrical circuits



- micro-gyroscope for navigation systems



Model order reduction (MOR) in optimization context

Dynamical system:

$$\begin{aligned} A(\omega,\gamma)x &= f \\ y &= d^T x \\ g &= \int_{\omega_{\min}}^{\omega_{\max}} |y|^2 d\omega \quad \text{(energy)} \end{aligned}$$



- Objective: minimize $g(\gamma)$
- Reduce the cost of computing y and ∇y:
  - Model reduction using moment matching = Padé approximation via Krylov methods
  - Allows for cheap computation of g and $\nabla_\gamma g$
- Assume uni-modal objective function

Outline

1. Motivation
2. MOR: Krylov-Padé methods
3. Parametric MOR
4. Trust region approach
5. Block Krylov method for low rank parameters
6. Conclusions

Left and right Krylov spaces

- For the linear case: $A(\omega) = I - \omega B$:
- Right Krylov space:

$$f, Bf, B^2 f, \ldots, B^{k-1} f$$

  Basis: $V_k = [v_1, \ldots, v_k]$
- Left Krylov space:

$$d, B^* d, B^{2*} d, \ldots, B^{(k-1)*} d$$

  Basis: $W_k = [w_1, \ldots, w_k]$
- Reduced model:

$$\begin{aligned} \widehat{A}(\omega)\widehat{x} &= \widehat{f} \\ \widehat{y} &= \widehat{d}^T \widehat{x} \end{aligned}$$

with $\widehat{A} = W_k{}^* A V_k$, $\widehat{f} = W_k{}^* f$, $\widehat{d} = V_k{}^* d$.

Moment matching

- Let $A(\omega) = I - \omega B$
- System output: $y = d^T (I - \omega B)^{-1} f$.

Theorem

Define the expansions:

$$\begin{aligned} y &= y_0 + \omega y_1 + \omega^2 y_2 + \cdots \\ \widehat{y} &= \widehat{y}_0 + \omega \widehat{y}_1 + \omega^2 \widehat{y}_2 + \cdots \end{aligned}$$

Let V_k and W_k be right and left Krylov basis resp., then

$$y_j = \widehat{y}_j \quad \textit{with} \quad j = 0, 1, \ldots, 2k-1 \ .$$

- Rational (Padé) approximation:

$$y(\omega) = \sum_{j=1}^{n} \frac{\rho_j}{\lambda_j - \omega} \qquad \widehat{y}(\omega) = \sum_{j=1}^{k} \frac{\widehat{\rho}_j}{\widehat{\lambda}_j - \omega}$$

- poles are Ritz values (approximate eigenvalues)

Gradient

- Build two-sided reduced model for fixed value of γ for
- $y = d^T A(\omega, \gamma_0)^{-1} f$
- Gradient:

$$\frac{dy}{d\gamma} = \left(A(\omega, \gamma_0)^{-*} d\right)^* \left(\frac{dA(\omega, \gamma_0)}{d\gamma}\right) \left(A(\omega, \gamma_0)^{-1} f\right)$$

- Blue part can be computed from right-Krylov space:

$$A(\omega, \gamma_0)^{-1} f \approx V_k(\widehat{A}^{-1}(\omega, \gamma_0)\widehat{f})$$

(k moments matched for $A(\omega, \gamma_0)^{-1} f$.)

- Red part can be computed from left-Krylov space:

$$A(\omega, \gamma_0)^{-*} d \approx W_k(\widehat{A}^{-*}(\omega, \gamma_0)\widehat{d})$$

(k moments matched for $A(\omega, \gamma_0)^{-*} d$.)

Gradient

- Full gradient

$$\frac{dy}{d\gamma} = \left(A(\omega,\gamma_0)^{-*}d\right)^* \left(\frac{dA(\omega,\gamma_0)}{d\gamma}\right) \left(A(\omega,\gamma_0)^{-1}f\right)$$

- Reduced gradient

$$\begin{aligned}\frac{d\widehat{y}}{d\gamma} &= \left(W_k\widehat{A}(\omega,\gamma_0)^{-*}\widehat{d}\right)^* \left(\frac{dA(\omega,\gamma_0)}{d\gamma}\right) \left(V_k\widehat{A}(\omega,\gamma_0)^{-1}\widehat{f}\right) \\ &= \left(\widehat{A}(\omega,\gamma_0)^{-*}\widehat{d}\right)^* \left(\frac{d\widehat{A}(\omega,\gamma_0)}{d\gamma}\right) \left(\widehat{A}(\omega,\gamma_0)^{-1}\widehat{f}\right)\end{aligned}$$

- $\frac{d\widehat{y}}{d\gamma}$ and $\frac{dy}{d\gamma}$ match k moments. [Antoulas, Beattie, Gugercin 2010] [Yue, M. 2011]
- Almost free computatation of the gradient, regardless the number of parameters!

Implementation issues

- Linear case: $A(\omega) = A_0 + \omega A_1$.

$$(A_0 + \omega A_1)x = f$$

- Replace by:

$$\begin{aligned} (I + \omega B)x &= b \quad \text{with} \quad B = A_0^{-1}A_1, b = A_0^{-1}f \\ y &= d^T x \end{aligned}$$

- Krylov space: sequence of matrix vector multiplies with B
- Computational cost:
  - Sparse matrix factorization of A_0
  - k sparse matrix vector products with A_1
  - k backward solves with A_0.

Nonlinear frequency dependence

$$(K + i\omega C - \omega^2 M)x = f$$

- 'Linearization':
  - Define matrices A and B

$$A = \begin{bmatrix} K & \\ & I \end{bmatrix} \qquad B = \begin{bmatrix} iC & -M \\ I & \end{bmatrix}$$

so that

$$(A - \omega B)\begin{pmatrix} x \\ \omega x \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

This is called a linearization, a similar trick as the solution of second order ODE's.

Linearizations

- Higher order polynomials

$$(A_0 + \omega A_1 + \cdots + A_p \omega^p)x = f$$

- Methods based on Companion 'linearization' [Amiraslani, Corless, Lancaster, 2009]
- Transform to

$$\left(\begin{bmatrix} A_0 & A_1 & \cdots & A_{p-1} \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} - \omega \begin{bmatrix} 0 & \cdots & \cdots & -A_p \\ I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ \omega x \\ \vdots \\ \omega^{p-1} x \end{bmatrix} = \begin{matrix} 0 \\ \\ \\ 0 \end{matrix}$$

- Can also be used for truly nonlinear problems (using Taylor expansion) [Van Beeumen, M., Michiels 2012]

Nonlinear output

- Quadratic output:

$$\begin{aligned}(A_0 - \omega A_1)x &= f \\ y &= x^* S x\end{aligned}$$

  with S a low rank matrix.

- reduced model:

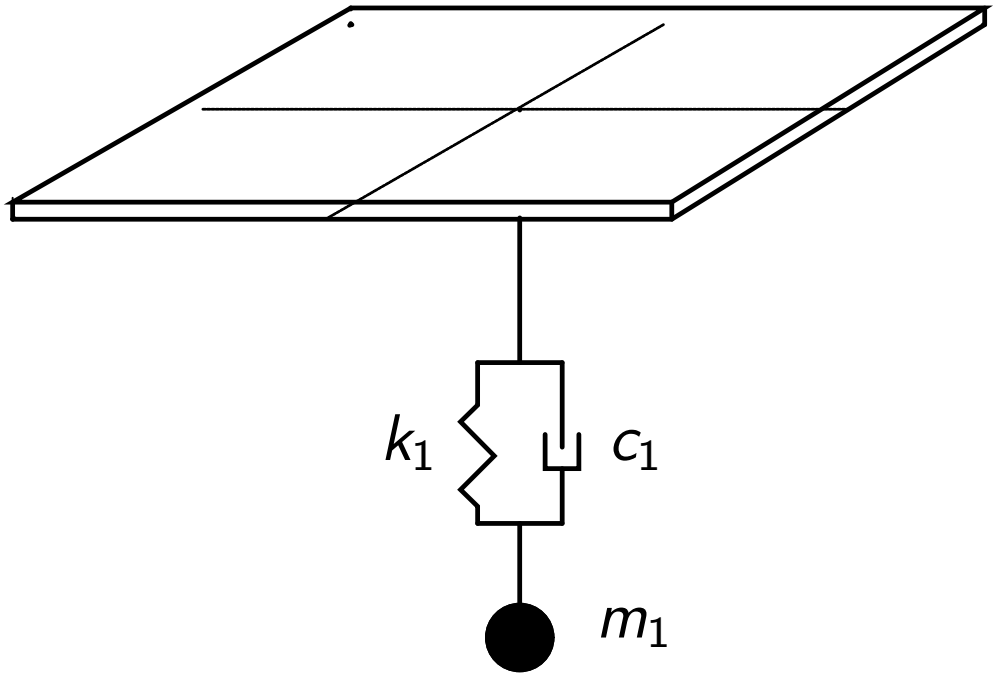
$$\begin{aligned}(\widehat{A}_0 - \omega \widehat{A}_1)\widehat{x} &= \widehat{f} \\ y &= \widehat{x}^* \widehat{S} \widehat{x}\end{aligned}$$

  with $\widehat{A}_0 = W^T A_0 V$, $\widehat{A}_1 = W^T A_1 V$, $\widehat{S} = V^T S V$ and $\widehat{f} = W^T f$.

- V: Krylov space with matrix $A_0^{-1} A_1$ and starting vector $A_0^{-1} f$
- W: Block-Krylov space with matrix $A_0^{-1} A_1$ and SV.
- Matching between $k + \sqrt{k}$ and $k + k$ moments
  [Van Beeumen, Van Nimmen, Lombaert, M., 2012]

Design optimization

- Determination of optimal parameters of a vibrating system
- Example: optimal parameters for a damper of a floor in a building near a noisy road



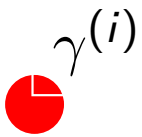
Design optimization

- Parametrized linear system:

$$\begin{cases} (K(\gamma) + i\omega C(\gamma) - \omega^2 M(\gamma))x &= f \\ y &= d^T x \end{cases}$$

- Find parameters γ so that
  - $\|y\|_2 = \int_0^{\omega_{\max}} |y|^2 d\omega$ is minimal
  - $\|y\|_\infty = \sup_0^{\omega_{\max}} |y|^2$ is minimal
- Assume: uni-modal, non necessarily smooth
- Expensive evaluation of y and the gradient

Overview



- Use MOR for function and gradient evaluation

Algorithm

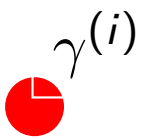
- We use the Damped BFGS optimization method
- On iteration i:
  - Build reduced model for $\gamma^{(i)}$ g
  - Compute g and ∇g from the reduced model
- Objective may not be smooth: use sufficient decrease condition and possibly backtracking after the BFGS step

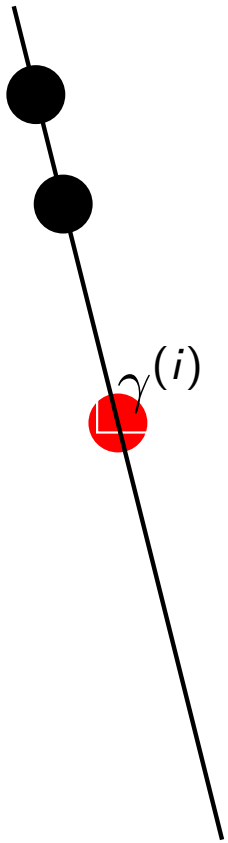
Numerical examples

- Floor with damper ($n = 29800$)
- Reduced model $k = 7$
- Determine optimal parameters c_1 and k_1

	Direct method	MOR
Matrix size	29800	7
Optimizer computed	$(12231609, 106031.18)$	$(12231614, 106031.22)$
Function value	$1.316093349\,10^{10}$	$1.316093349\,10^{10}$
CPU time	7626s	179s

Overview





- Use MOR for function and gradient evaluation
- Use PMOR for line search optimization + backtracking

Parametric MOR: PIMTAP

- Reduced model for equations of the following form:

$$\begin{aligned}(G_0 + \lambda G_1 + s(C_0 + \lambda C_1))x &= b \\ y &= d^* x\end{aligned}$$

- Perform moment matching, i.e. terms associated with $s^i \lambda^j$.
- Algorithms
  - Many, many papers, e.g. Lihong Feng.
  - PIMTAP [Li, Bai, Su, Zeng 2007], [Li, Bai, Su, Zeng 2008], [Li, Bai, Su 2009]
  - Subspace that contains the vectors:

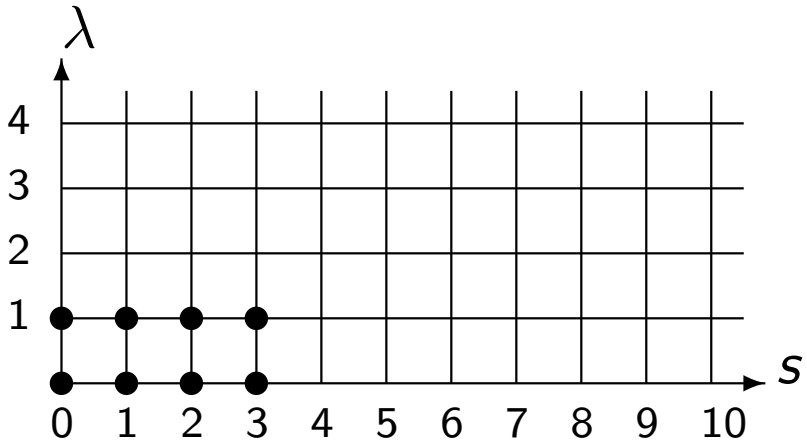
$$\begin{aligned}r_i^j &= 0 \quad i < 0, \quad \text{or} \quad j < 0 \\ r_0^0 &= G_0^{-1} b \\ r_i^j &= -G_0^{-1}(C_0 r_{i-1}^j + G_1 r_i^{j-1} + C_1 r_{i-1}^{j-1})\end{aligned}$$

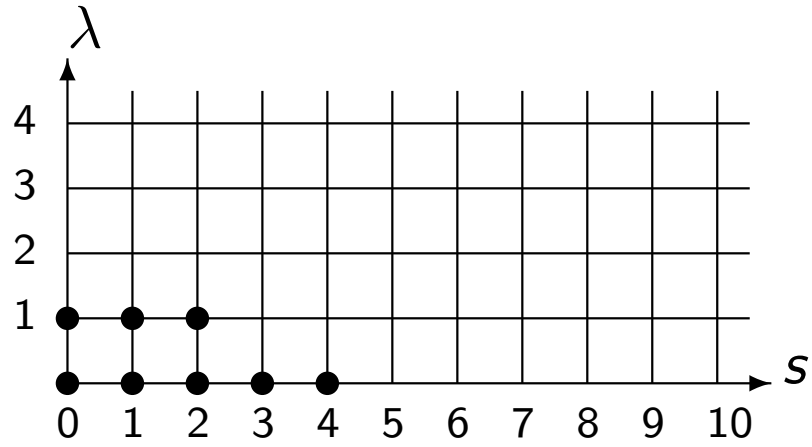
    These are the moments in the expansion $x \sim \sum r_i^j s^i \lambda^j$

- Two-sided PIMTAP
  - One PIMTAP for b and one for d.

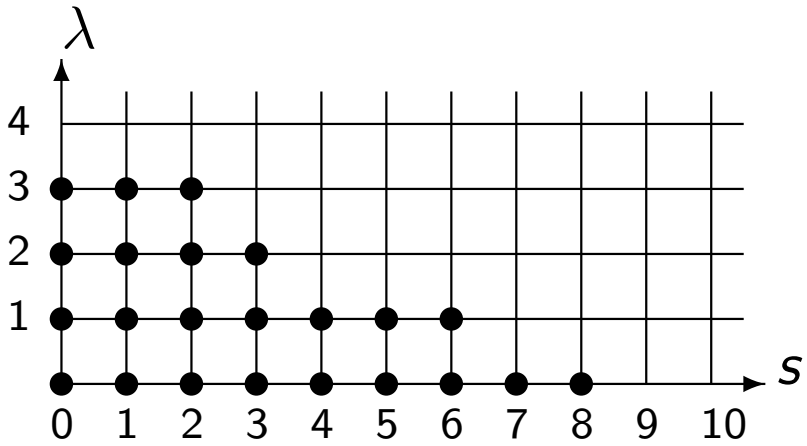
Moment matching of the gradient

- Moment matching patterns for left and right Krylov spaces



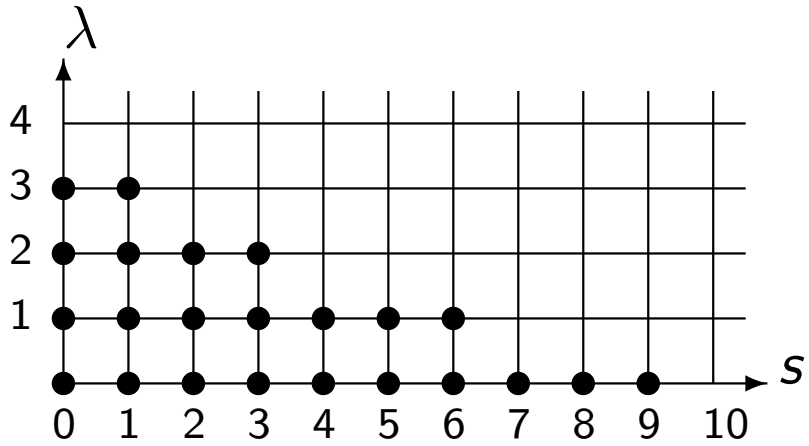


- Moment matching pattern for y

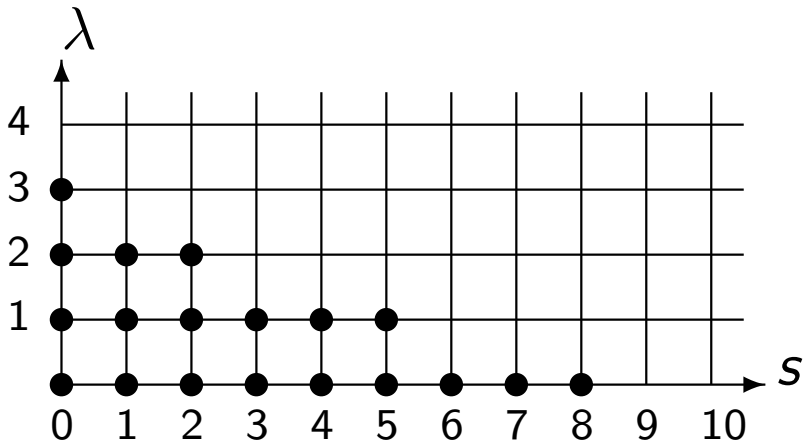


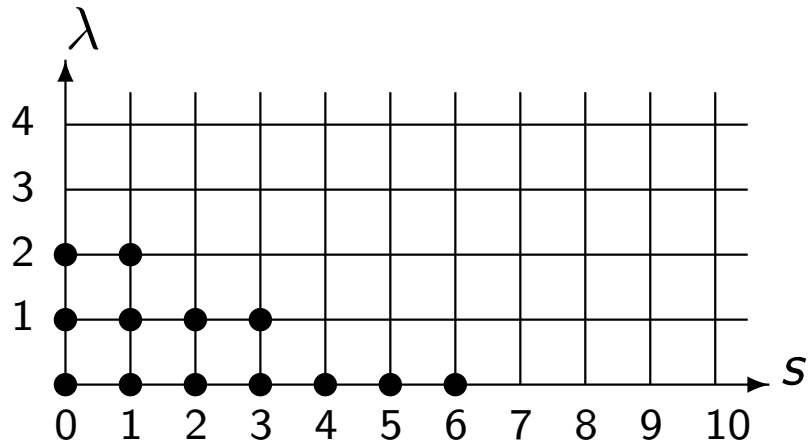
Moment matching of the gradient

- Moment matching pattern for y



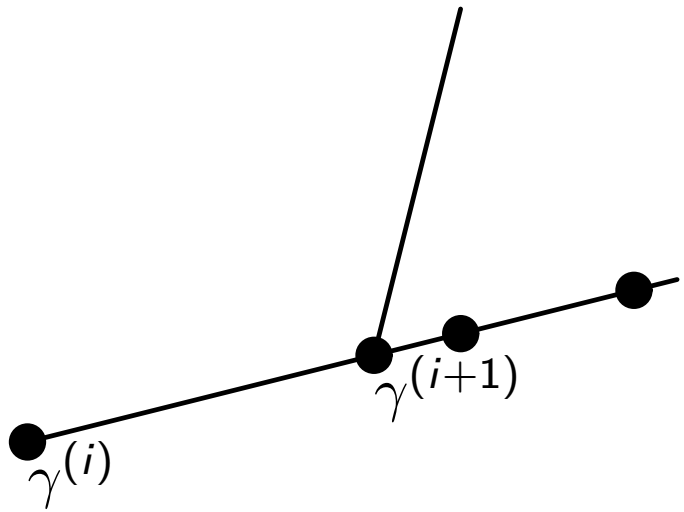
- Moment matching pattern for $\frac{\partial y}{\partial s}$ and $\frac{\partial y}{\partial \lambda}$





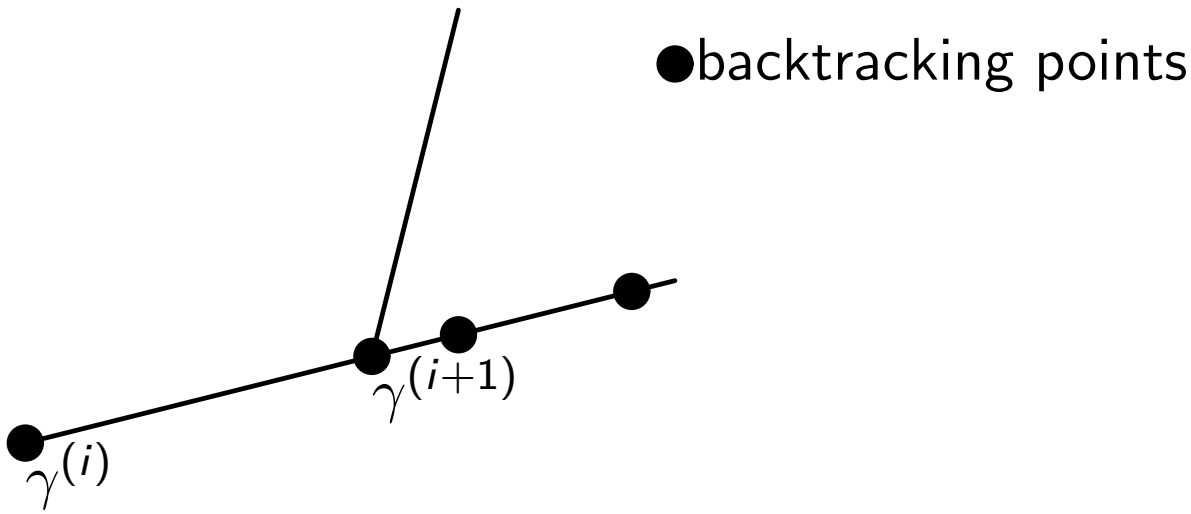
The MOR/PMOR Framework

- We can make a reduced model for ω and all parameters γ
- This is usually expensive and overkill: only reduce on the important directions
- So, we use PIMTAP for efficient line search:



  - The MOR Framework generates a reduced model for each γ accessed.
  - The PMOR Framework generates a reduced model for each line search iteration.

PMR framework

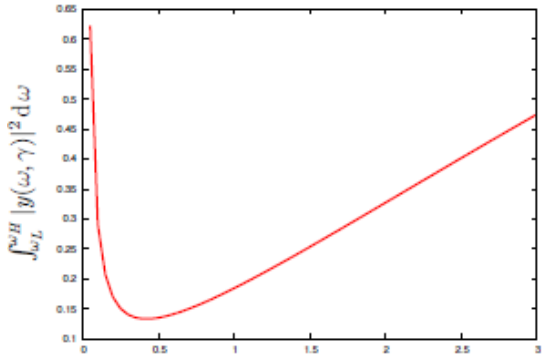


- For backtracking using Armijo line searcher:
  - the MOR Framework is better for smooth objective functions (The first trial point is often accepted for Quasi-Newton methods);
  - the PMOR Framework is better for non-smooth objective functions.
- For smooth objective functions, we can use exact line searches using the PMOR Framework.

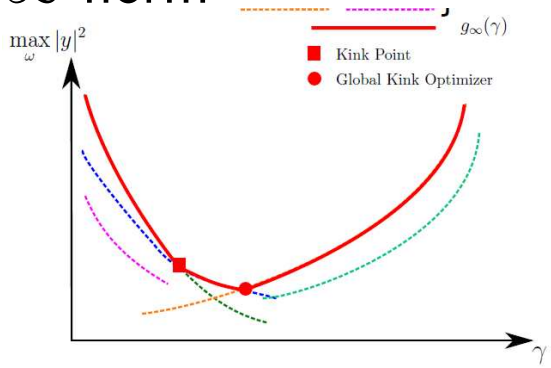
Example: min-max optimization

- Floor optimization problem (2 unknowns)
- Univariate objective function:

2 norm



∞ norm



- Comparison of MOR Framework and PMOR Framework

	2 norm			∞ norm		
	iter	k	Time	iter	k	Time
direct	11 (+6)	29,800	7626	15 (+108)	29,800	41,069
MOR	13 (+8)	7	179	15 (+120)	7	1,104
PMOR	12 (+4)	7+3	360	14 (+90)	7+3	417

Example: exact line search optimization

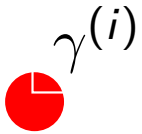
- Lamot bridge finite element model ($n = 25,962$)
- The goal is to determine the optimal stiffness and damping coefficient of four bridge dampers (=8 parameters).



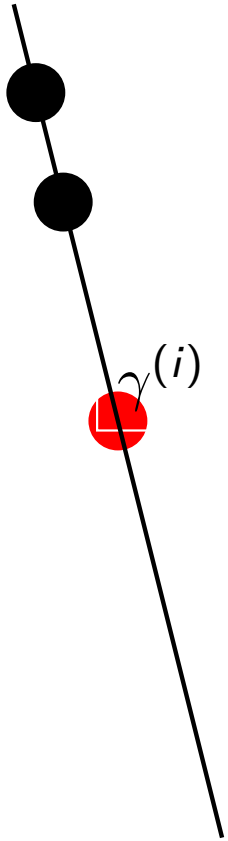
- Numerical results

	MOR	(backtracking) PMOR	Exact line search PMOR
order	12	11+4+2+1	11+4+2+1
iterations	70	73	27
time (s)	879	1830	703

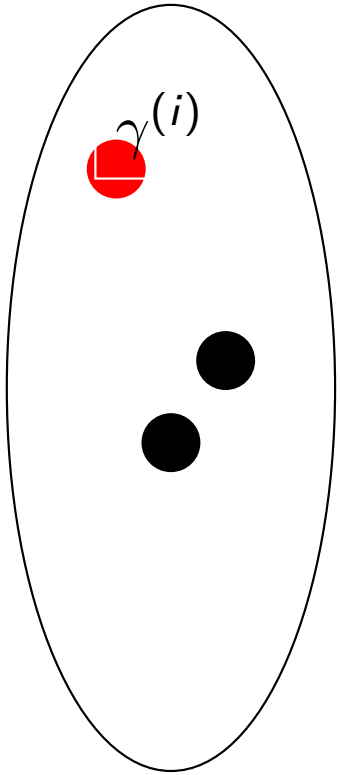
Overview



- Use MOR for function and gradient evaluation



- Use PMOR for line search optimization + backtracking



- Use interpolatory MOR for trust region optimization

Trust region approach

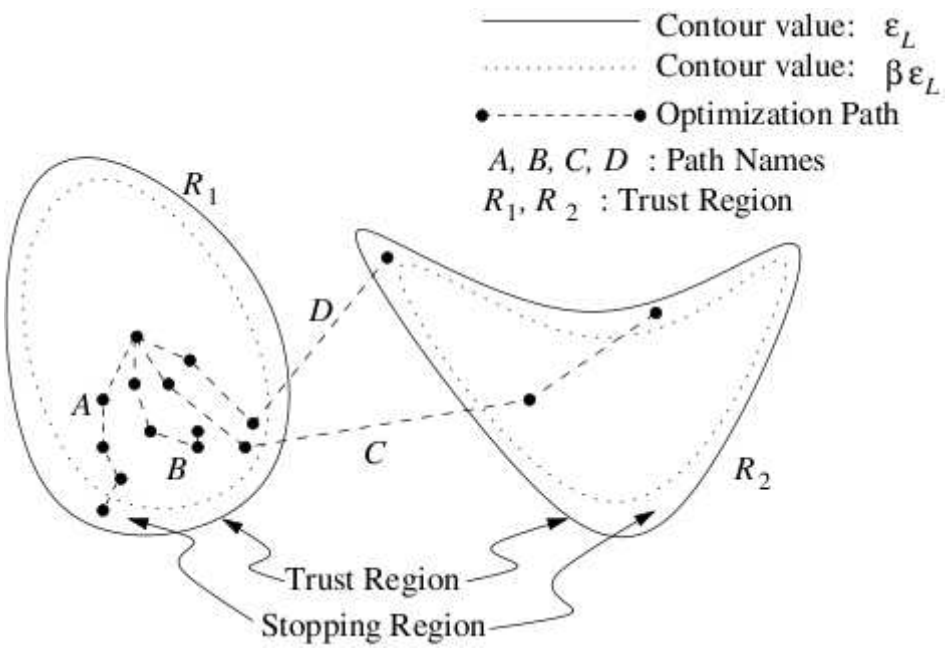
- The reduced model interpolates the exact function and the gradient in one point,
- so, we expect the model to be useful in a region around the interpolation point.
- On the ith iteration:
  1. Build Krylov space for $\gamma = \gamma^{(i)}$
  2. Use the subspace to make a reduced model in ω and γ:

$$\begin{aligned} \widehat{A}(\omega,\gamma)\widehat{x} &= \widehat{f} \\ \widehat{y} &= \widehat{d}^T\widehat{x} \end{aligned}$$

     with $\widehat{A} = W_k{}^* A V_k$, $\widehat{f} = W_k{}^* f$, $\widehat{d} = V_k{}^* d$.
  3. Find region where the error is acceptable (= trust region)
- Convergence test rewritten in terms of reduced models (that is what you have)
- Build new model only when required from convergence theory

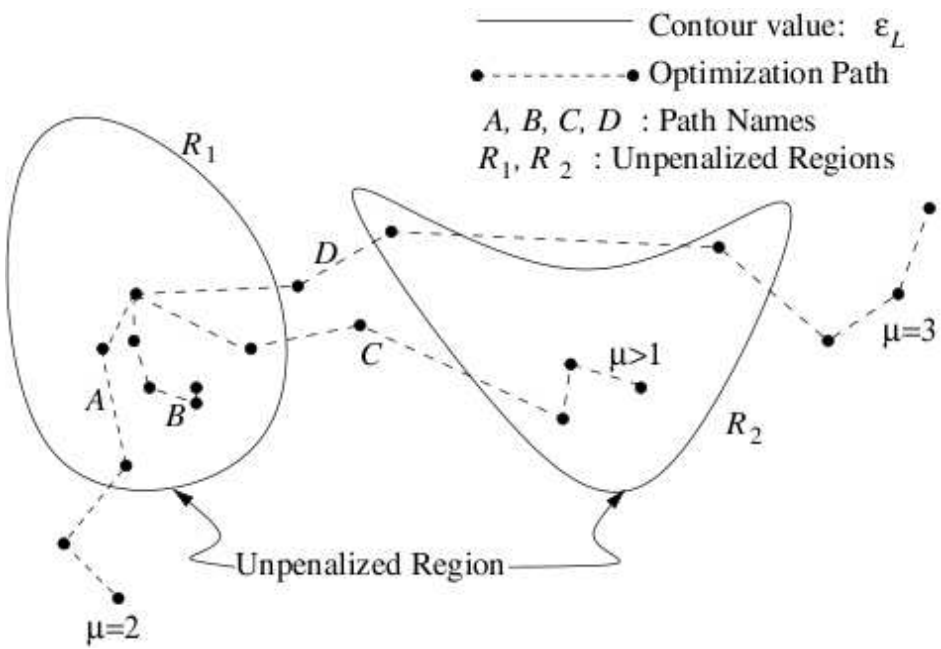
Subproblems

ETR



Terminate if close to boundary.

EP



Terminate if w is active for μ successive steps.

Example of the Lamot footbridge



- Lamot bridge finite element model ($n = 25,962$)
- The goal is to determine the optimal stiffness and damping coefficient of four bridge dampers (=8 parameters).

- Computation times:
  - without reduced modeling: 545 for one function evaluation, 70 function evaluations needed.
  - MOR Framework: 879 sec.
  - ETR: 200 sec.
  - EP: 189 sec.

Block Krylov method for low rank parameters

- Parametric system



$$\begin{aligned}(K - \omega^2 M + BC(\omega,\gamma)B^T)x &= f \\ y &= \varphi(x)\end{aligned}$$

where B is low rank r

- Reduced model:

$$\begin{aligned}(\widehat{K} - \omega^2 \widehat{M} + \widehat{B}C(\omega,\gamma)\widehat{B}^T)\widehat{x} &= \widehat{f} \\ y &= \varphi(V\widehat{x}) \\ \widehat{K} = V^T K V \qquad & \widehat{M} = V^T M V \\ \widehat{B} = V^T B \qquad & \widehat{f} = V^T f\end{aligned}$$

Block Krylov method for low rank parameters

- Reduced model:

$$
\begin{aligned}
(\widehat{K} - \omega^2 \widehat{M} + \widehat{B} C(\omega, \gamma) \widehat{B}^T)\widehat{x} &= \widehat{f} \\
y &= \varphi(V\widehat{x}) \\
\widehat{K} = V^T K V \qquad & \qquad \widehat{M} = V^T M V \\
\widehat{B} = V^T B \qquad & \qquad \widehat{f} = V^T f
\end{aligned}
$$

  with V Krylov basis for
  - Use block Krylov method:
  - starting block: $K^{-1}[f, B]$.
  - matrix: $K^{-1}M$
- Approximation properties:
  - For any γ, $2k$ moment match for ω
  - γ dependence is fully held by C and therefore represented exactly.
- We need only one reduced model for all optimization steps.

Example

2-norm Optimization of the Footbridge Damper Optimization Problem

	Nr Models	Order	Optimum	CPU Time
The MOR Framework	70	12	24.77751651	879s
The PMOR Framework	27	18	24.77751661	703s
ETR	3	20	24.78594112	205s
EP	3	20	24.7775166	190s
Block Arnoldi	1	30	24.77751815	20.4s

- Objective at the initial point: 142.34188.
- Damped-BFGS converges in 71 iterations.
- A single evaluation of g costs 540s.

Conclusions

- We proposed three type of methods to accelerate design optimization with (P)MOR.
- All of them are efficient for accelerating design optimization.
- Performance:

  Block Arnoldi $>$ *ETR/EP* $>$ *The (P)MOR Framework*

- Applicability:

  The MOR Framework, ETR/EP $>$ *The PMOR Framework, Block Arnoldi*

- Reliability:

  Depends on error estimation for the reduced model

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