Overview of methods for algebraic model order reduction

Karl Meerbergen

KU Leuven

Numerical Linear Algebra

1. Motivation
2. Modal approximation
3. Padé Krylov methods
4. Balanced truncation

Examples of large scale systems

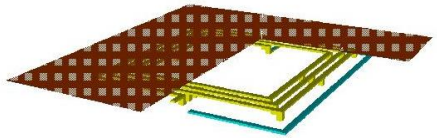
- Planes



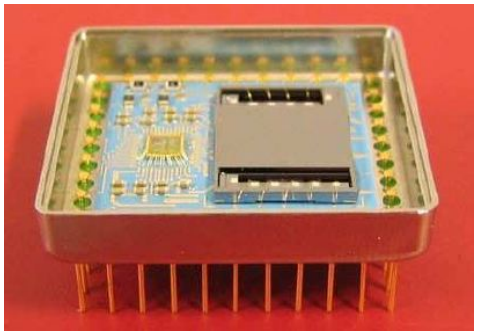
- Footbridge (= lightweight construction)



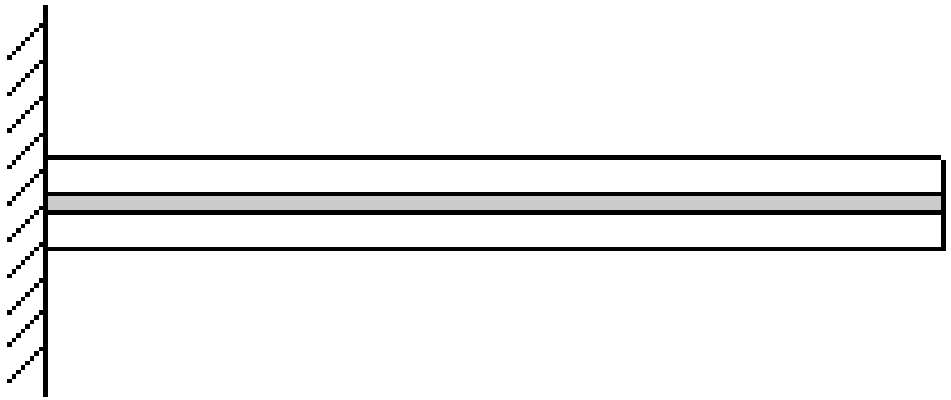
- Maxwell-equation – electrical circuits



- micro-gyroscope for navigation systems



Nonlinear damping



- Clamped sandwhich beam
- 168 degrees of freedom (finite elements)
- Nonlinear (non-polynomial) in the frequency

$$K_e x + \frac{G_0 + G_\infty (is\tau)^\alpha}{1 + (is\tau)^\alpha} K_v - s^2 M x = f$$

with $\alpha = 0.675$ and $\tau = 8.230$.

Definition

- Given a linear time invariant system (LTI)

$$\begin{aligned} \frac{dx}{dt} &= Ax + fu(t) \\ y &= c^* x + \delta u(t) \end{aligned}$$

- $u(t)$: input, $y(t)$: output
- state vector $x \in \mathbb{R}^n$ with very large n.
- Usually, it is assumed $\delta = 0$.
- Goal: determine

$$\begin{aligned} \frac{d\hat{x}}{dt} &= \hat{A}\hat{x} + \hat{f}u(t) \\ \hat{y} &= \hat{c}^* \hat{x} + du(t) \end{aligned}$$

  with $\hat{x} \in \mathbb{R}^k$ where $k \ll n$ is such that $\|y - \hat{y}\|$ is small.
- Sometimes, also systems in discrete time

$$\begin{aligned} x_{k+1} &= Ax_k + fu_k \\ y_k &= c^* x_k + \delta u_k \end{aligned}$$

Frequency domain

- Laplace transform X

$$\begin{aligned} x(0) &= 0 \quad \text{(initial value)} \\ sX &= AX + fU(s) \\ Y &= c^*X \end{aligned}$$

- We use x also in the Laplace domein instead of X

$$x(s) = (sI - A)^{-1} fu(s)$$

- Transfer function H: $y = Hu$.

$$H = c^*(sI - A)^{-1} f$$

- H is a rational function with eigenvalues of A as poles
- The goal of MOR: approximate H by a rational polynomial of low degree

$$\hat{H} = \hat{c}^*(sI - \hat{A})^{-1} \hat{f}$$

Frequency domain

- Often, the error on an approximation of $H(s)$ is expressed in the H-infinity norm:

$$\|H - \hat{H}\|_{\mathcal{H}_\infty} = \sup_{\omega} |H(i\omega) - \hat{H}(i\omega)|$$

Applications

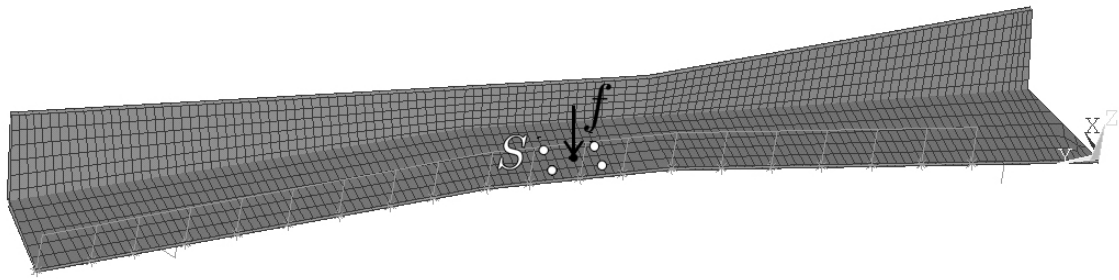
- Electrical circuits ($n = 200 \rightarrow k = 5, \ldots, 20$)
- Mechanical vibrations, acoustics, structures: ($n = 10^4, \ldots, 10^6 \rightarrow k = 20, \ldots, 1000$)
- Fluid dynamics ($n \gg 10^6$)

Example

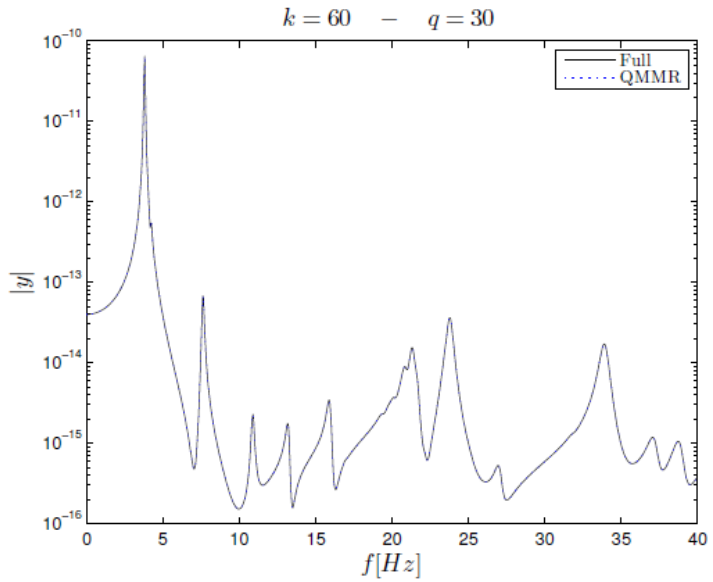
- Lamot footbridge problem:

$$\begin{aligned}(K + i\omega C - \omega^2 M)x &= f \\ y &= x^* S x\end{aligned}$$





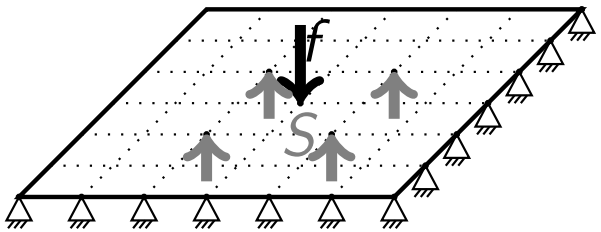
- Waterwegen en Zeekanaal NV, Emotec/Emergo-Group, Herbosch-Kiere, Flemish Government
- Full model: $n = 25,962$
- Reduced model for $\omega \in [0, 40]$Hz: $m = 60$



Example

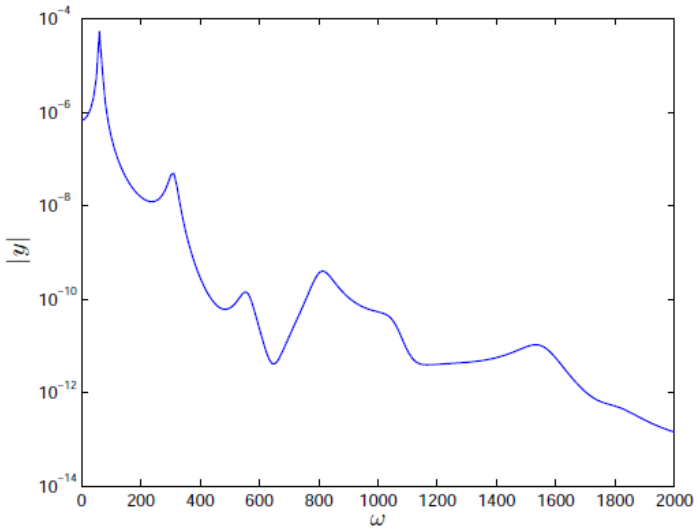
- Concrete floor problem:

$$\begin{aligned}(K + i\omega C - \omega^2 M)x &= f \\ y &= x^* S x\end{aligned}$$



- Full model: $n = 29,799$
- Reduced model for $\omega \in [0, 2000]\text{rad}/s$: $m = 40$
- Cost for evaluating y for p values of ω:

p	Direct method	MOR (Krylov)
1000	1,750.00	7.54
10	17.50	7.54



Methods

- Twosided methods SISO (single-input single-output)
  - Modal approximation ('Dominant pole algorithm' — Krylov-methods)
  - Padé-approximation (rational approximation via Krylov-methods)
  - Balanced truncation
  - In the time domein: POD (proper orthogonal decomposition)
- Singlesided methods: solution of a linear system with a parameter

$$\begin{aligned}(sI - A)x &= f \\ (I - sA)x &= f\end{aligned}$$

- Currently
  - a large research effort on problems with parameters (=PMOR):

$$H = c(p)^*(sI - A(p))^{-1}f(p)$$

    with $p \in \mathbb{C}^m$ parameters.
  - models with nonlinear s dependency:

$$H = c^*(A_0 + sA_1 + e^{-\tau s}A_2))^{-1}f$$

Modal approximation/ modal truncation

- Given

$$A = U\Lambda W^* \quad , \quad W^*U = I \quad , \quad \Lambda = \text{diag}(\lambda_1, \lambda_n)$$

- transfer function:

$$\begin{aligned} H(s) &= c^*(sI - U\Lambda W^*)^{-1} f \\ &= (c^*U)(sI - \Lambda)^{-1}(W^*f) \\ &= \sum_{j=1}^{n} \frac{(c^*u_i)(w_i^*f)}{s - \lambda_i} \end{aligned}$$

$$R_i = (c^*u_i)(w_i^*f) \quad = \text{ 'residue'}$$

Modal approximation/modal truncation

- transfer function:

$$\begin{aligned} H(s) &= (c^* U)(sI - \Lambda)^{-1}(W^* f) \\ &= \sum_{j=1}^{n} \frac{(c^* u_i)(w_i^* f)}{s - \lambda_i} \end{aligned}$$

- Truncation to k terms

$$\hat{H}(s) = \sum_{j=1}^{k} \frac{(c^* u_i)(w_i^* f)}{s - \lambda_i}$$

  with $k \ll n$.

- $\lambda_1, \ldots, \lambda_k$ are chosen so that

$$\frac{R_1}{s - \lambda_1}, \ldots, \frac{R_k}{s - \lambda_k}$$

  are dominant terms (dominant $\rightarrow$ peaks in the transfer function)

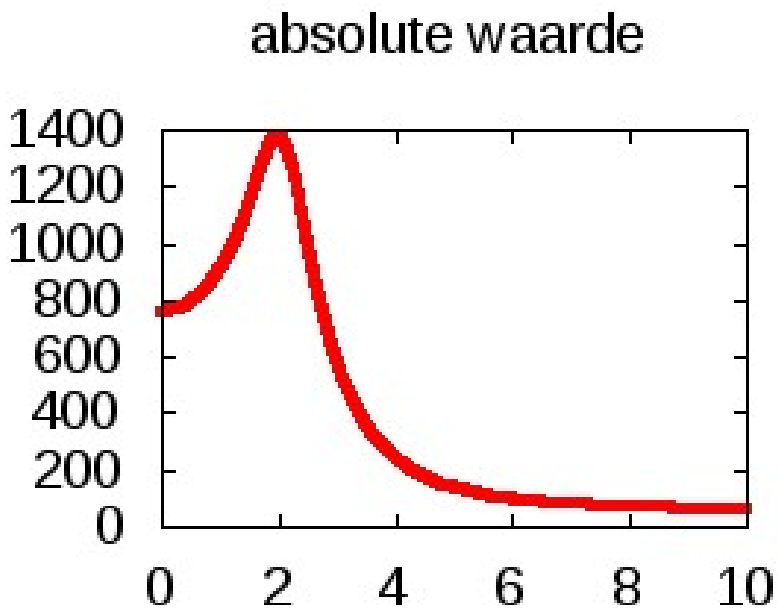
Numerical example

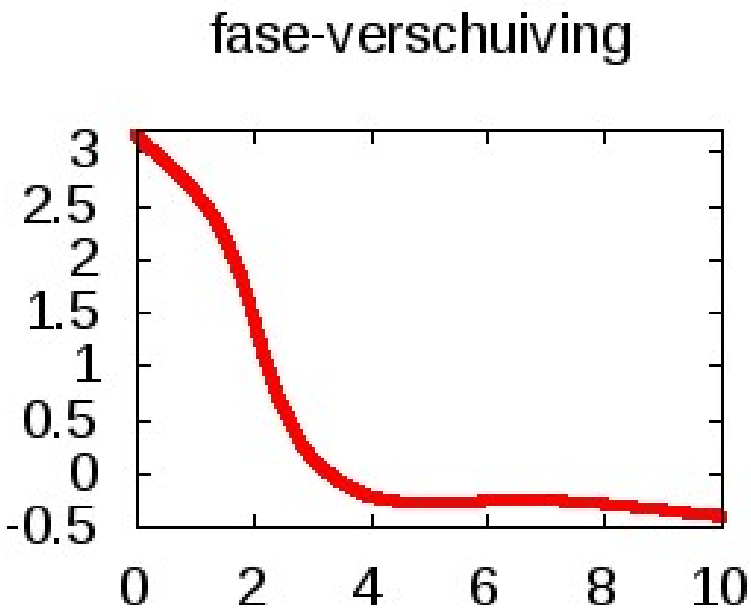
- Olmstead equation

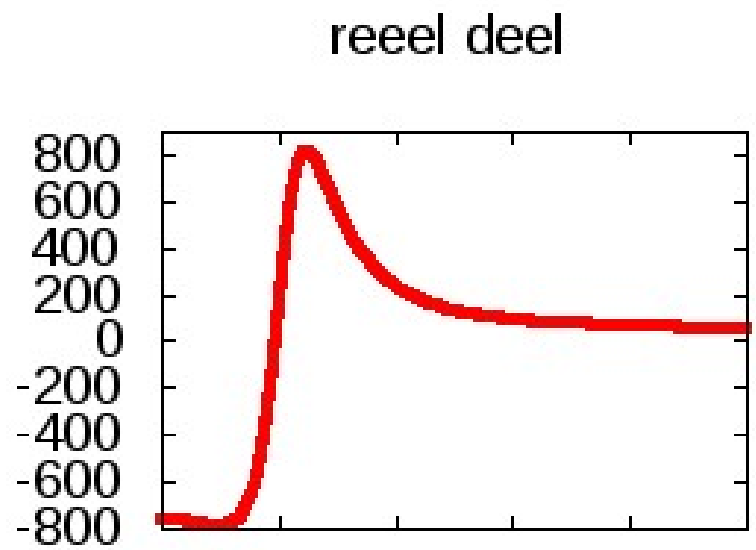
$$\begin{aligned} \frac{\partial u}{\partial t} &= (1-\gamma)\frac{\partial^2 v}{\partial X^2} + \gamma\frac{\partial^2 u}{\partial X^2} + \rho u - u^3 \\ \beta\frac{\partial v}{\partial t} &= u - v \end{aligned}$$

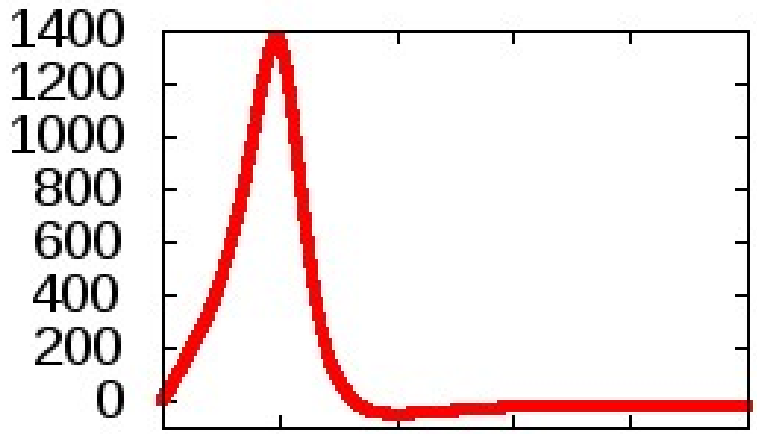
- u represents the speed of the fluid
- v is related to visco-elastic forces.
- homogeneous Dirichlet boundary conditions
- Spatial discretization: $n = 1000$.

Numerical example: transfer function



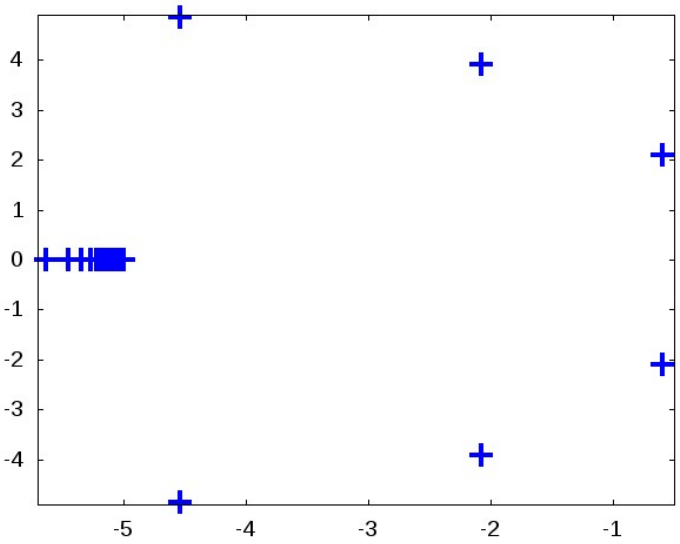




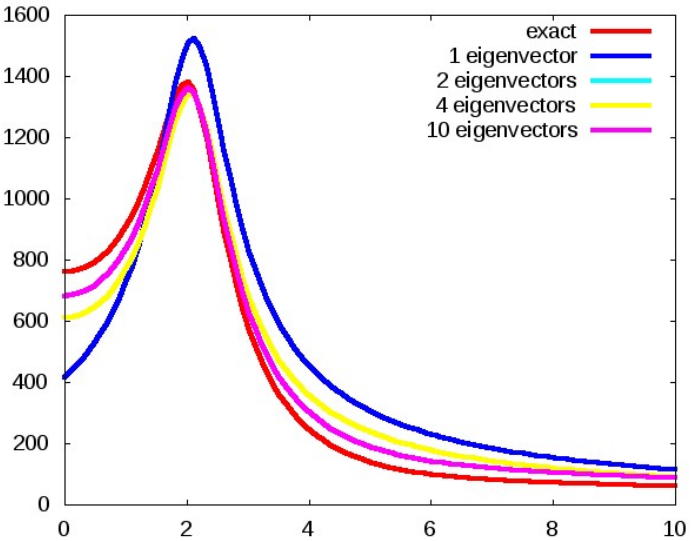


Numerical example

- Part of the spectrum:



- Modulus of the transfer function for $k = 1, 2, 4, 10$.



Computation of dominant poles

- Krylov-methods (see further)
- Newton-method $\rightarrow$ dominant pole algorithm
  - transfer function: $H(s) = c^*(sI - A)^{-1}f$
  - pole of $H(s)$ is zero of $Q(s) = H^{-1}(s)$.
  - Newton-method:

$$\begin{aligned} s_{k+1} &= s_k - \frac{Q(s_k)}{Q'(s_k)} \\ &= s_k + \frac{H(s_k)}{H'(s_k)} \qquad (Q' = -H'H^{-2}) \\ &= s_k - \frac{c^* x_k}{y_k^* x_k} = s_k - \frac{y_k^* f}{y_k^* x_k} \end{aligned}$$

with

$$\begin{aligned} x_k &= (sI - A)^{-1}f \qquad (H' = -c^*(sI - A)^{-2}f) \\ y_k &= (sI - A)^{-*}c \end{aligned}$$

Computation of dominant poles

- Newton-method:

$$s_{k+1} = s_k - \frac{c^* x_k}{y_k^* x_k} = s_k - \frac{y_k^* f}{y_k^* x_k}$$

with

$$\begin{aligned} x_k &= (sI - A)^{-1} f \\ y_k &= (sI - A)^{-*} d \end{aligned}$$

- Note for simple eigenvalue λ_1:

$$\lim_{s \to \lambda_1} x_k = \text{right eigenvector}$$
$$\lim_{s \to \lambda_1} y_k = \text{left eigenvector}$$

Example

- Dominant pole algorithm for Olmstead equation
- Starting value $s = 0.1 + i$ (must be complex!)

k	s_k	$\|Ax_k - s_k s_k\| / \|s_k x_k\|$
1	$-1.2357 + 1.7846i$	0.63943
2	$-0.34438 + 2.01865i$	0.19616
4	$-0.59389 + 2.08036i$	0.090048
5	$-0.59359 + 2.10557i$	0.0080219
6	$-0.59348 + 2.10537i$	$7.5568\,10^{-5}$
7	$-0.59348 + 2.10537i$	$6.7182\,10^{-9}$

- Convergence to a dominant pole is not guaranteed.

Dominant pole algorithm with subspace projection

- The dominant pole algorithm converges to one pole only
- Convergence is not guaranteed
- Combine with subspace projection:
  - Increase chances of computing dominant pole
  - Compute more than one pole

1: Selection initial value of pole: s
2: Let right basis be $V = []$ and left basis be $W = []$.
3: **while** No convergence of desired number of poles **do**
4: Solve $(sI - A)x = f$ and $(sI - A)^* y = c$
5: Add x to the subspace: $V = \text{orth}([V, x])$.
6: Add y to the subspace: $W = \text{orth}([W, y])$.
7: Project: $\widehat{A} = W^* A V$
8: Compute eigenvalues of $\widehat{A} \rightarrow \lambda$
9: Select s among the eigenvalues λ.
10: **end while**

Solving parametric linear systems

- Solve

$$(I - sA)x = f$$

  for many values of s around 0.
- Idea: use a Krylov subspace (Arnoldi's method = FOM)
  - Krylov space: $\{f, Af, A^2f, \ldots, A^{k-1}f\}$
  - Krylov vectors $V_k = [v_1, \ldots, v_k]$
  - Hessenberg matix $H_k = V_k^* A V_k$
  - Recurrence relation: $AV_k - V_k H_k = r_k e_k^*$ with $V_k^* r_k = 0$
- We now approximate x by $\widetilde{x} = V_k z$ with

$$(I - sH_k)z = \|f\|_2 e_1$$

- s is close to zero, $I - sA \approx I$ and FOM converges quickly

Solving parametric linear systems

Three interpretations of Arnoldi method:

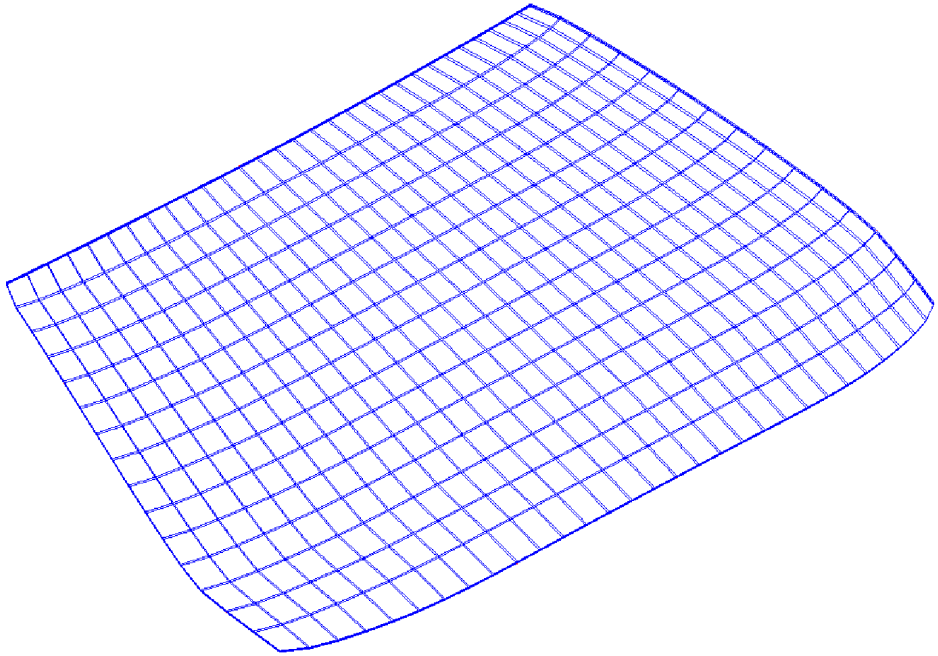
- Iterative method on $I - sA$ (FOM)
- Padé approximation (moment matching)
- Modal approximation

Example

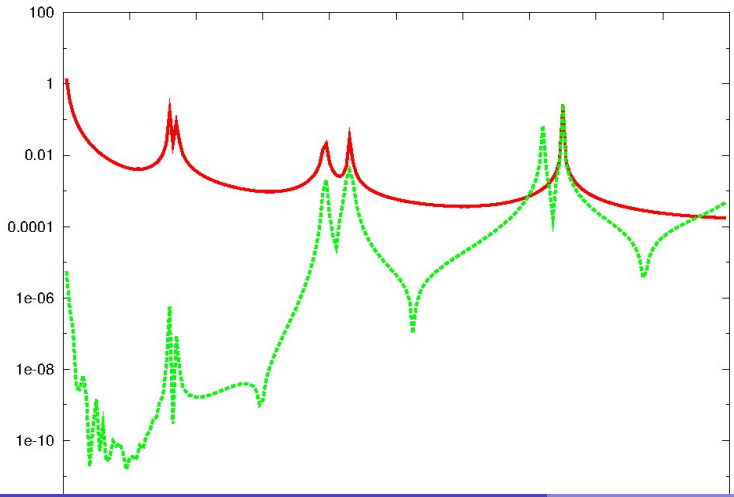
- 3D model of a BMW windscreen
- System:

$$(K-\omega^2 M)x = f \quad K, M \in \mathbb{R}^{n\times n}$$

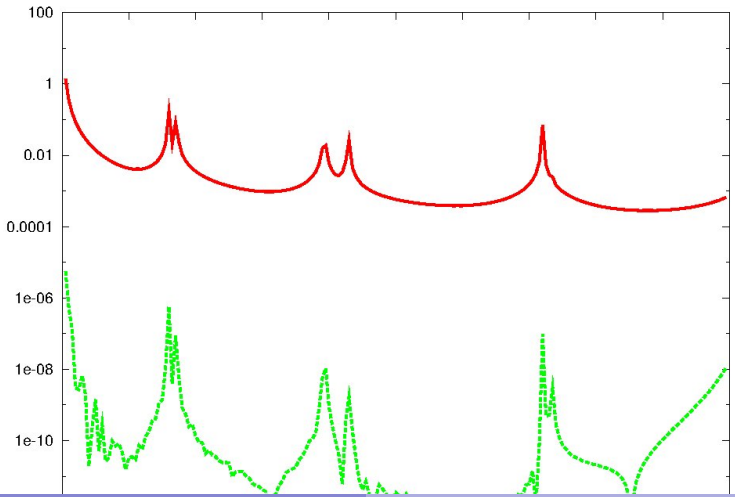
- $n = 22,692$
- Frequency range $\omega \in [0, 100]$



$k = 10$ vectors



$k = 20$ vectors



Moment matching

$$\begin{aligned}(I - sA)x &= f \\ (I - sH_k)z &= \|f\|_2 e_1\end{aligned}$$

- Develop x as $x = x_0 + sx_1 + s^2x_2 + \cdots$ assuming the series converges in a region around 0.
  - From $(I - sA)(x_0 + sx_1 + s^2x_2 + \cdots) = f$, we have:
  - $x_0 = f$
  - $x_1 - Ax_0 = 0 \longrightarrow x_1 = Ax_0$
  - $x_j - Ax_{j-1} = 0 \longrightarrow x_j = Ax_{j-1} = A^j f$

Moment matching

$$\begin{aligned}(I - sA)x &= f \\ (I - sH_k)z &= \|f\|_2 e_1\end{aligned}$$

- Develop $x = x_0 + sx_1 + s^2x_2 + \cdots$:
  - $x_j = Ax_{j-1} = A^j f$
- Develop z as $z = z_0 + sz_1 + s^2z_2 + \cdots$ assuming the series converges in a region around 0.
  - From $(I - sH_k)(z_0 + sz_1 + s^2z_2 + \cdots) = \|f\|e_1$, we have:
  - $z_0 = \|f\|e_1$
  - $z_1 - H_k z_0 = 0 \longrightarrow z_1 = H_k z_0$
  - $z_j - H_k z_{j-1} = 0 \longrightarrow z_j = H_k z_{j-1} = \|f\| H_k^j e_1$

Moment matching

$$\begin{aligned}(I - sA)x &= f \\ (I - sH_k)z &= \|f\|_2 e_1\end{aligned}$$

- Develop $x = x_0 + sx_1 + s^2x_2 + \cdots$:
  - $x_j = Ax_{j-1} = A^j f$
- Develop $z = z_0 + sz_1 + s^2z_2 + \cdots$
  - $z_j = H_k z_{j-1} = \|f\| H_k^j e_1$

  Arnoldi:

$$AV_k - V_k H_k = r_k e_k^*$$

- We have that $x_j = V_k z_j$ for $j = 1, \ldots, k-1$:
  - $x_0 = f = \|f\| v_1 = V_k(\|f\| e_1)$
  - By construction $v_k^* A^j f = 0$ for $j < k$.
  - Induction: z_{j-1} has last $k - j$ components equal to zero, so $e_k^* z_{j-1} = 0$ for $j \leq k$
  - Induction: $x_j = Ax_{j-1} = AV_k z_j = V_k H_k z_{j-1} + r_k e_k^* z_{j-1} = V_k H_k z_{j-1}$

Padé via Krylov

- Linear SISO model:

$$\begin{aligned} (I - sA)x &= f \\ H &= c^* x \end{aligned}$$

- Two Krylov-spaces

$$\begin{aligned} \mathcal{K}_k(A, f) &= \text{span}\{f, Af, A^2 f, \ldots, A^{k-1} f\} \to V_k \\ \mathcal{K}_k(A^T, c) &= \text{span}\{c, A^T c, A^{2T} c, \ldots, A^{(k-1)T} c\} \to W_k \end{aligned}$$

- Biorthogonal bases: $W_k^* V_k = I$
- Reduced model:

$$\begin{aligned} (I - s\widehat{A})z &= W_k^* b \quad \text{with} \quad \widehat{A} = W_k^* A V_k \\ \widehat{H} &= c^* V_k z \end{aligned}$$

- Model order is k

Padé via Krylov

- Moment matching:
  - Recurrence relations:

$$\begin{aligned} AV_k - V_k H_k &= r_k e_k * \\ AW_k - W_k T_k &= t_k e_k * \end{aligned}$$

  - With $W_k^* V_k = I$ and $\widetilde{A} = W_k^* A V_k$:
  - Matching k moments for $x(s) = (I - sA)^{-1} f$
  - Matching k moments for $t(s) = (I - sA)^{-T} c$
  - Matching $2k-1$ moments for $H(s) = c^*(I - sA)^{-1} f$ and $\widehat{H} = \widehat{c}^*(I - s\widehat{H})^{-1}\widehat{f}$

$$\begin{aligned} H_{k+j-1} &= c^* A^{k-1-j} f = (A^{(k-1)*} c)^* (A^j f) \\ &= \|f\|\|c\| (W_k \widehat{A}^{(k-1)*} e_1)^* (V_k \widehat{A}^j e_1) \\ &= \|f\|\|c\| e_1^* \widehat{A}^{k-1+j} e_1 = \widehat{H}_{k+j-1} \end{aligned}$$

$j = 0, \ldots, k-1.$

Padé via Krylov

- Padé approximation:

$$\widehat{H}(s) = \frac{\alpha_1}{s-\mu_1} + \cdots + \frac{\alpha_k}{s-\mu_k}$$

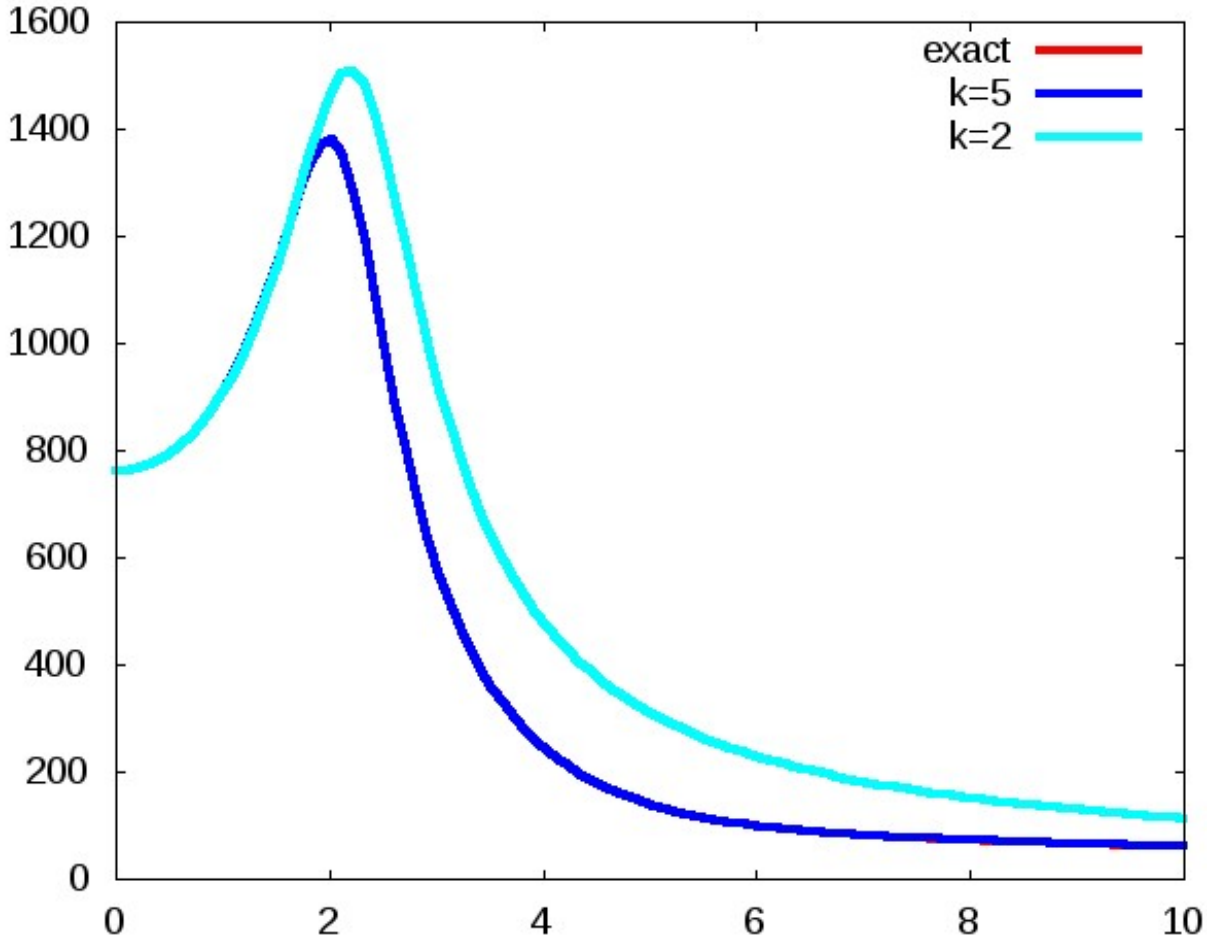
- The poles are the eigenvalues of $W_k^* A V_k$.
- Eigenvalues of $W_k^* A V_k$ are Ritz values (or better Petrov-Galerkin values) of the oblique projection:

$$W_k^*(A - \lambda I) V_k z = 0$$

- So, we have a connection with modal truncation.

Example

- Krylov models of orders 2 and 5



PDE constraint optimization: example



- Lamot footbridge FEM model
- Goal is to optimize demper parameters k_1 and c_1.

$$\min_{k_1,c_1} \int_{\omega_L}^{\omega_H} |y(\omega; k_1, c_1)|^2 d\omega$$

Methods:

- BFGS applied to the full system
- BFGS applied to the reduced model. Each iteration a new reduced model is built.
- BFGS with exact line search on a reduced parametric model.
- BFGS on a reduced model with penalty function derived from an error estimate (EP).
- Block Krylov method that produces reduced model for the entire parametric domain.

Timings for footbridge optimization

	Nr Models	Order	Optimum	CPU Time
The MOR Framework	70	12	24.77751651	879s
The PMOR Framework	27	18	24.77751661	703s
EP	3	20	24.7775166	190s
Block Arnoldi	1	30	24.77751815	20.4s

A single evaluation of g costs 540s.

Balanced truncation

- Time invariant system

$$\begin{aligned} \frac{dx}{dt} &= Ax + fu(t) \\ y &= c^* x \end{aligned}$$

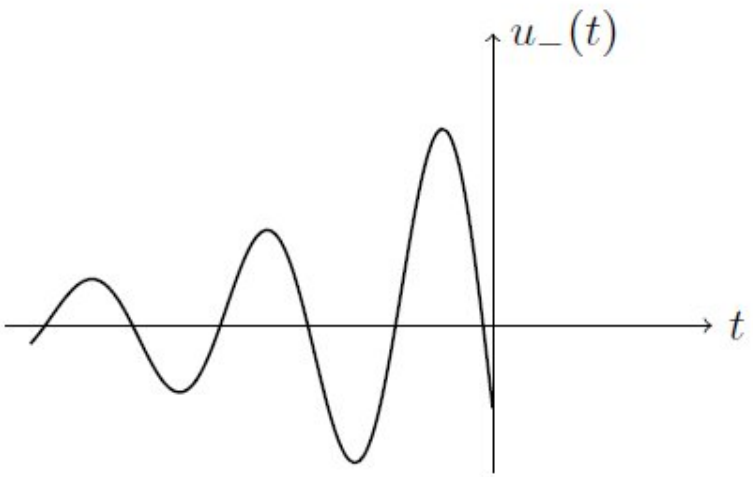
- $u(t) = 0$ for $t > 0$.
- output

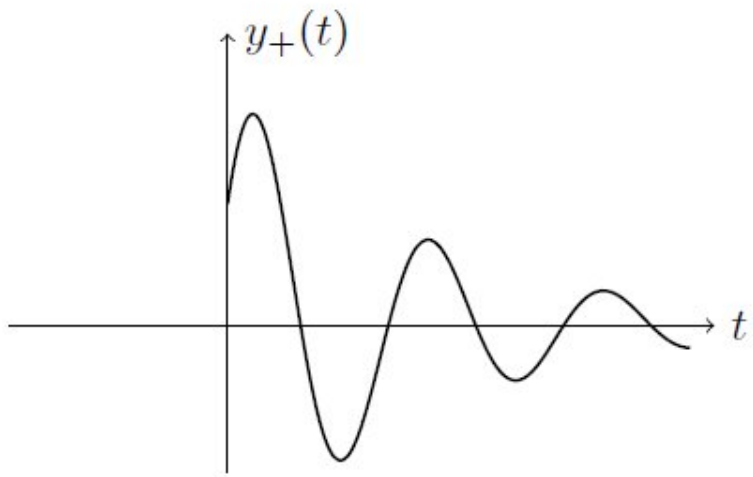
$$\begin{aligned} y(t) &= \int_{-\infty}^{0} c^* e^{(t-\tau)A} fu(\tau) d\tau \\ &= c^* e^{tA} \int_{0}^{+\infty} e^{\tau A} fu(-\tau) d\tau \\ &= c^* e^{tA} x(0) \end{aligned}$$

- The solution can be seen as time integration from $t = 0$ given an initial value $x(t = 0)$, determined by the output in $(-\infty, 0)$.

Balanced truncation

- $y(t) = c^* e^{tA} x(0)$
- The solution can be seen as time integration from $t = 0$ given an initial value $x(t = 0)$, determined by the output in $(-\infty, 0)$.





Connection with Krylov-spaces

- Consider a system in discrete time

$$\begin{aligned} x_{k+1} &= Ax_k + fu_k \\ y_k &= c^* x_k + \delta u_k \end{aligned}$$

- output

$$\begin{aligned} y_k &= \sum_{-\infty}^{0} c^* A^{k-j} f u_j \\ &= c^* A^k \sum_{0}^{+\infty} A^j f u_{-j} = c^* A^k x_0 \end{aligned}$$

- Define

$$\mathcal{C} = [f,\ Af,\ A^2 f, \ldots] \qquad \mathcal{O} = \begin{bmatrix} c^* \\ c^* A \\ c^* A^2 \\ \vdots \end{bmatrix}$$

Hankel-operator

- $x_0 = \mathcal{C}[u_{-1}, u_{-2}, \ldots]^*$
- $[y_1, y_2, \ldots] = \mathcal{O}x_0$
-
$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} = \mathcal{O}\mathcal{C} \begin{pmatrix} u_{-1} \\ u_{-2} \\ \vdots \end{pmatrix}$$
- $\mathcal{H} = \mathcal{O}\mathcal{C}$ is the Hankel-operator.
- Goal: find a low degree approximation of $\mathcal{H}$.
- Observability Gramian (time continuous systeem):
$$G_o = \mathcal{O}^*\mathcal{O} = \sum_0^{+\infty} (c^* A^j)^* (c^* A^j)$$
- Controllability Gramian:
$$G_c = \mathcal{C}\mathcal{C}^* = \sum_0^{+\infty} (A^j f)(A^j f)^*$$
- Note: a Gramian is a matrix of the form WW^*.

Gramians

- Observability Gramian (time continuous systeem):

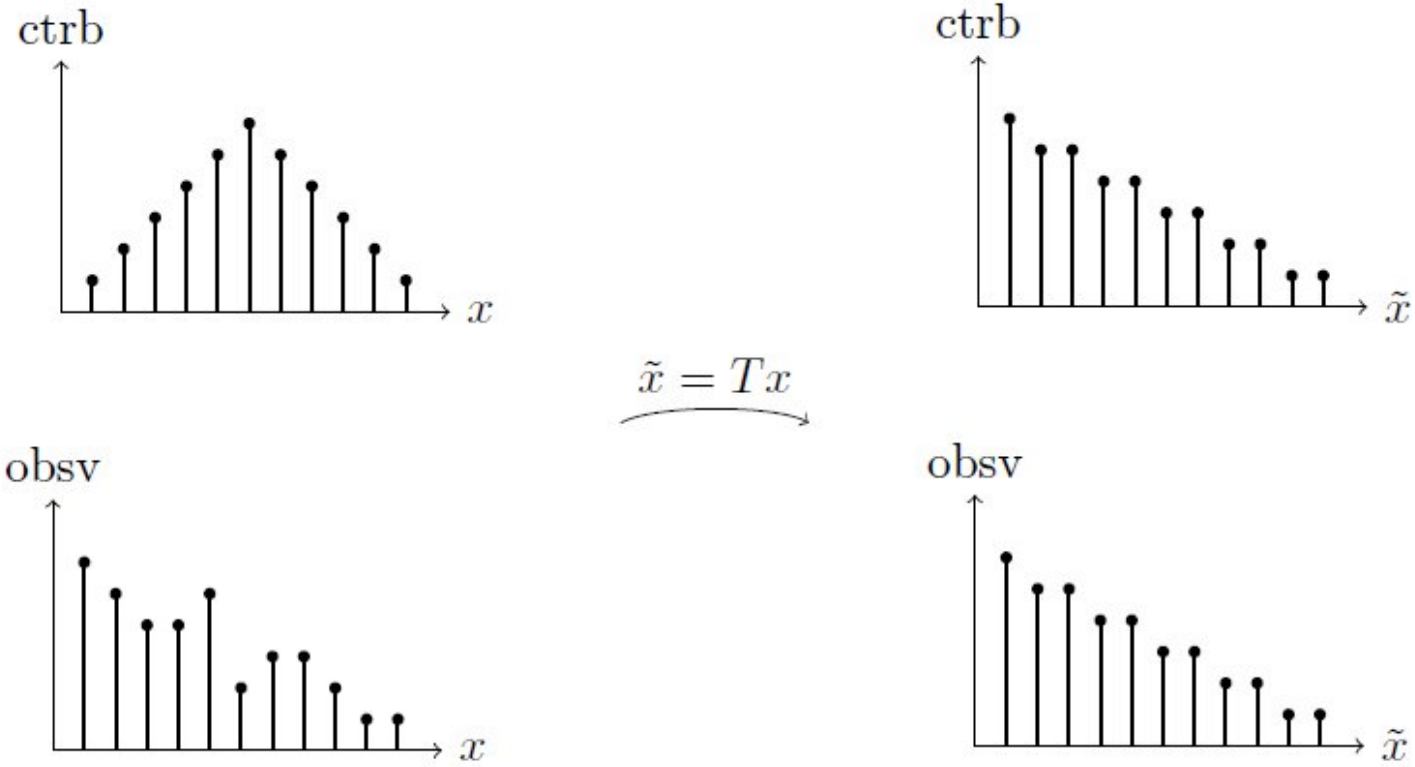
$$G_o = \mathcal{O}^*\mathcal{O} = \int_0^{+\infty} (c^* e^{At})^* (c^* e^{At}) dt$$

- Controllability Gramian:

$$G_c = \mathcal{C}\mathcal{C}^* = \int_0^{+\infty} (e^{At} f)(e^{At} f)^* dt$$

- Eigenvalues of $\mathcal{H}^*\mathcal{H}$ are eigenvalues of $G_o G_c = \mathcal{O}^*(\mathcal{O}\mathcal{C})\mathcal{C}^*$
- Singular values of $\mathcal{H}$ are diagonal elements of Σ: Hankel singular values

Balanced realization



- Transformation of variables $\tilde{x} = Tx$:

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= TAT^{-1}\tilde{x} + Tfu(t) \\ y &= c^* T^{-1}\tilde{x} \end{aligned}$$

Same eigenvalues, but other Gramians (diagonal)

Balanced realization

- Transformation of variables $\tilde{x} = Tx$:

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= TAT^{-1}\tilde{x} + Tfu(t) \\ y &= c^* T^{-1}\tilde{x} \end{aligned}$$

- transformed Gramians

$$\begin{aligned} \tilde{G}_o &= T^{-*} G_o T^{-1} \\ \tilde{G}_c &= T G_o T^* \end{aligned}$$

- Determine T so that $\tilde{G}_o = \tilde{G}_c = \Sigma^2$ (diagonal matrix)

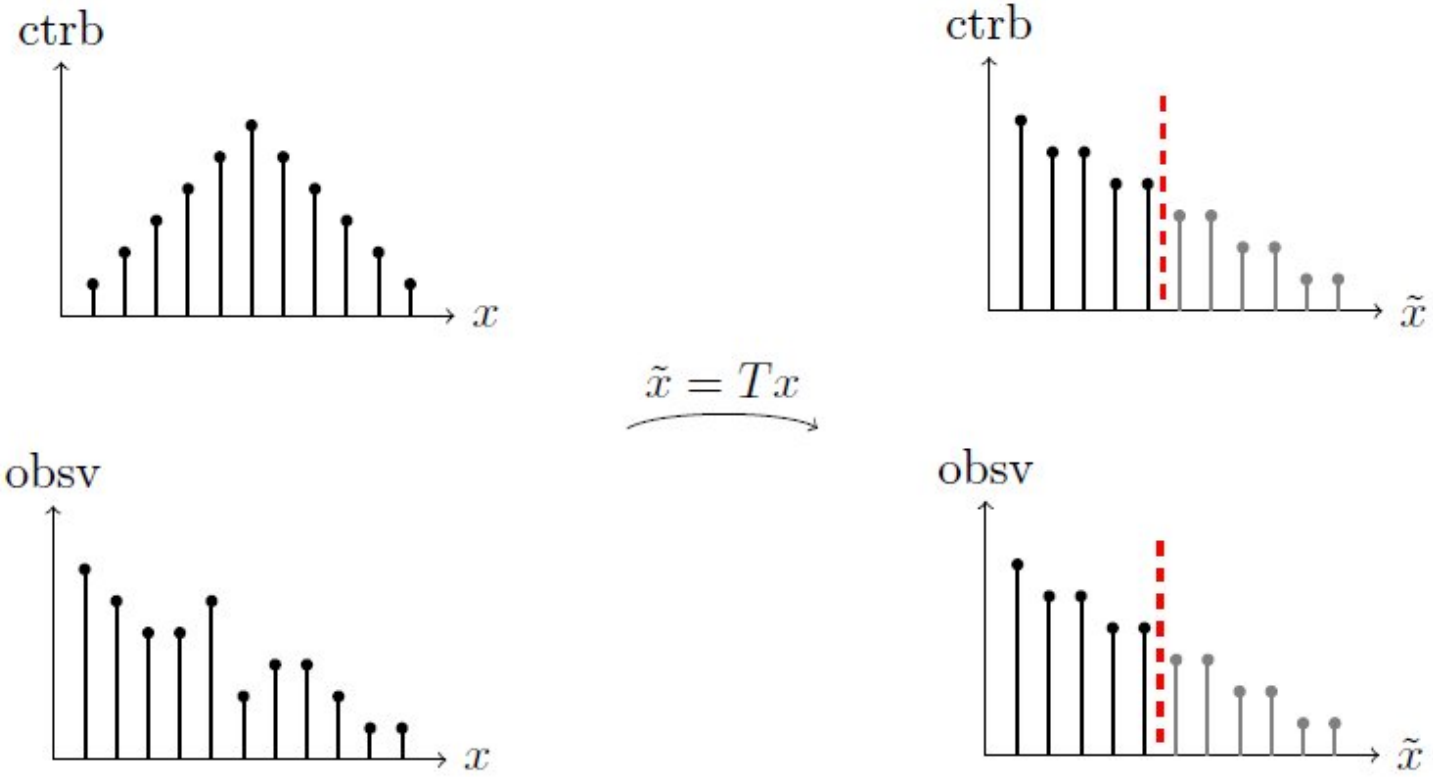
Balanced truncation

- Singular values of $\mathcal{H}$: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.
- Truncation: keep the k largest singular values
- Define $P = [I, 0]$ that selects the first k components of $\tilde{x}$.

$$\begin{aligned} \tilde{x} &= Tx \\ z &= P\tilde{x} = PTx \\ \frac{dz}{dt} &= PTAT^{-1}P^*z + PTf \\ \hat{H} &= c^*T^{-1}P^*z \end{aligned}$$

- $\|\hat{H} - H\|_{\mathcal{H}_\infty} \leq 2\sum_{j=k+1}^{n} \sigma_j$

Balanced truncation



- Truncation: keep the k largest singular values
- Transformation of variables $\tilde{x} = PTx$:

$$\begin{aligned} \frac{dz}{dt} &= PTAT^{-1}P^*z + PTfu(t) \\ y &= c^*T^{-1}P^*z \end{aligned}$$

Computation of PT: Cholesky factorisation

- The Gramians satisfy the following Lyapunov-equations:

$$\begin{aligned} AG_c + G_c A^* + ff^* &= 0 \\ A^* G_o + G_o A + cc^* &= 0 \end{aligned}$$

- If the linear system is stable, G_o and G_c are unique
- Compute Cholesky-factors of $G_o = UU^*$ and $G_c = LL^*$ (U plays the role of $\mathcal{O}^*$ and L the role of $\mathcal{C}$).
- Compute the SVD of U^*L. (singular values of $\mathcal{H}$): $U^*L = Z\Sigma Y^*$.
- Truncate the SVD: $U^*L \approx Z_1\Sigma_1 Y_1^*$
- Define

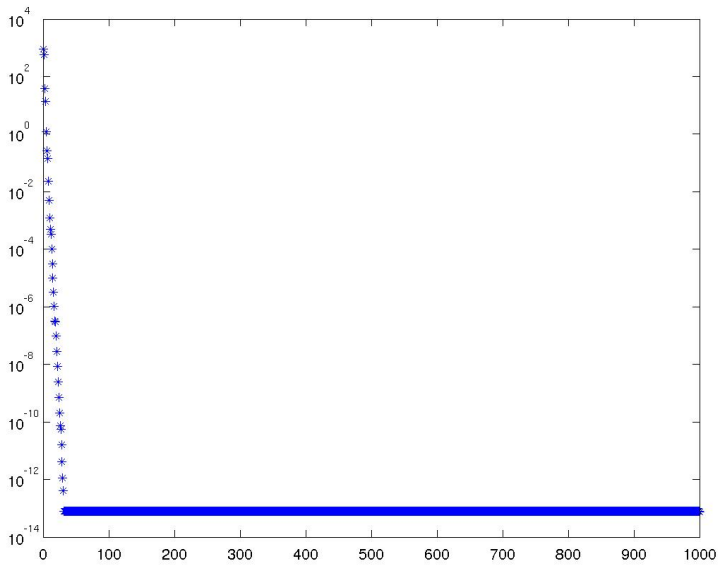
$$W_1 = LY_1\Sigma^{-1/2} \qquad V_1 = UZ_1\Sigma^{-1/2}$$

  $W_1^* V_1 = I$, so $V_1 W_1^*$ is a projector.
- Reduced model:

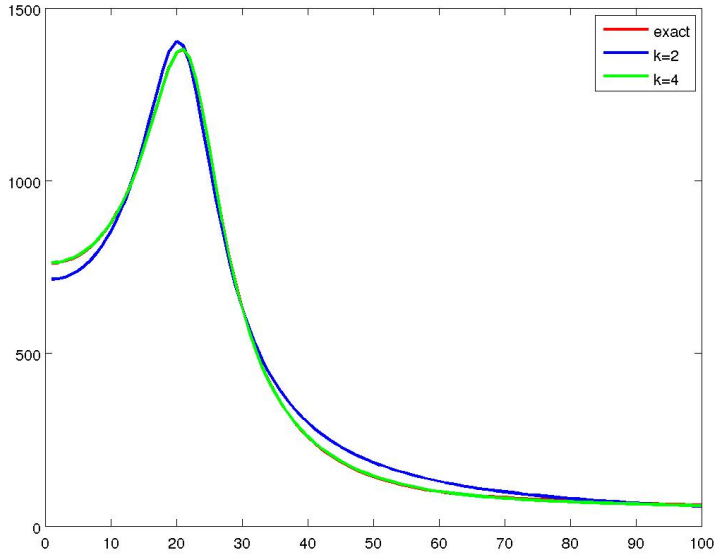
$$\begin{aligned} \frac{dz}{dt} &= W_1^* A V_1 z + W_1^* f \\ \hat{H} &= c^* V_1 z \end{aligned}$$

Example

- Hankel singular values quickly decay



- Approximations of orders 2 and 4.



Solving Lyapunov equations: Bartels-Stewart

- Lyapunov equation

$$AG + GA^* + bb^* = 0$$

- Schur decomposition of $A = USU^*$ with S upper triangular.
- Substitute in the Lyapunov equation:

$$\begin{aligned} USU^*G + GUS^*U^* + bb^* &= 0 \\ S(U^*GU) + (U^*GU)S^* + (U^*b)(U^*b)^* &= 0 \end{aligned}$$

- First solve $SZ + ZS^* + B = 0$ column by column:

1: **for** $k = n, n-1, \ldots, 1$ **do**
2: Solve the upper triangular system z_j from

$$(S + s_{j,j}I)z_j = -Be_j - \sum_{i=1}^{j-1} s_{i,j}z_i$$

3: **end for**
4: Let $Z = [z_1, \ldots, z_n]$.

- Then compute $G = UZU^*$
- Cost: $O(n^3)$.

POD

- Modelreduction in the time domain, also used for non-linear systems

$$\begin{aligned} \frac{dx}{dt} &= Ax + fu(t) \\ y &= c^* x \end{aligned}$$

- Make a matrix of snapshots: $X = [x(t_1), x(t_2), \ldots, x(t_k)]$.
- $XX^* = \sum_{j=1}^{k} x(t_j)x(t_j)^*$ is related to G_c in continuous time
- It is equally possible to make a basis Z for $[z(t_1), \ldots, z(t_k)]$ where

$$\frac{dz}{dt} = A^* z + c$$

- $ZZ^* = \sum_{j=1}^{k} z(t_j)z(t_j)^*$ is related aan G_o
- Make a reduced model like with balanced truncation.

Overview

- Balanced truncation:
  - Optimal (as SVD for a matrix): for a given tolerance, we find the smallest model with a prescribed error tolerance.
  - Solution of Lyapunov-equations required.
  - Requests a stable system
- Krylov-methods
  - Easy to use
  - Not optimal
  - Works for unstable systems
- Modal truncation
  - Suitable for symmetric matrices (purely imaginary eigenvalues)
  - In general, this is an expensive method
  - DPA (dominant pole algorithm) does guarantee the computation of the dominant poles
- Hot topic in research:
  - parametric MOR
  - Non-linear MOR

References


A.C. Antoulas.
Approximation of Large-Scale Dynamical Systems.
SIAM, Philadelphia, PA, USA, 2005.


P. Benner, V. Mehrmann, and D.C. Sorensen, editors.
Dimension Reduction of Large-Scale Systems.
Springer-Verlag, Berlin, Heidelberg, 2005.


W.H.A. Schilders, H.A. van der Vorst, and J. Rommes.
Model order reduction: theory, research aspects and applications,
volume 13 of *Mathematics in Industry*.
Springer, 2008.