Stability analysis of oscillatory systems subject to large delays: a synchronization point of view

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Abstract

The behavior of the rightmost characteristic roots of linear time-delay systems as a function of the delay parameter is studied for small values of a gain parameter. The results explain the qualitative behavior of the stability regions in the delay parameter space of oscillatory systems subjected to large delays, and shed a new light on the corresponding stabilization problem. In particular, for a chain of multiple oscillators the determination of a stabilizing value of the delay parameter is interpreted as a phase synchronization problem. The results are illustrated with the output feedback stabilization of an oscillator and the stability analysis of a gyroscopic system.

1 Introduction

We study the behavior of the zeros of the function

$$H(s;\ \tau,\epsilon) := f(s) + \epsilon g(s) e^{-s\tau}, \tag{1}$$

where $f:\ \mathbb{C}\rightarrow\mathbb{C}$ and $g:\ \mathbb{C}\mapsto\mathbb{C}$ are entire, as a function of the 'gain parameter' $\epsilon\in\mathbb{R}$ and the 'delay parameter' $\tau\geq 0$.

In the context of stabilization and control the problem under consideration is motivated by the stability analysis of the feedback interconnection of a single input, single output system with transfer function

$$\frac{Y(s)}{U(s)} = \frac{P(s)}{Q(s)} e^{-s\tau_1}, \tag{2}$$

and the controller

$$C(s) = \epsilon \frac{X(s)}{Y(s)} e^{-s\tau_2}. \tag{3}$$

The resulting closed-loop quasi-polynomial of (2) and (3) takes the form (1) provided

$$f(s) = Q(s)Y(s),\ g(s) = P(s)X(s),\ \tau = \tau_1 + \tau_2.$$

In [1] it was shown that delayed output feedback, where $C(s) = \epsilon e^{-s\tau_2}$, may stabilize the oscillator,

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + \Omega^2}, \tag{4}$$

while this is not possible using static output feedback. This illustrates that, although the existence of a time-delay in a feedback loop usually causes instability or performance degradation [15, 20, 28], there are situations where time-delays have a stabilizing effect. The characterization of such situations was the topic of the papers [19, 22], where necessary and sufficient conditions for the stabilizability of single input, single output systems using delayed output feedback were presented, along with a controller construction procedure. In [1, 22] it was illustrated that a small value of the controller gain may lead to the stabilization of (4) for particular large delay values and to the occurrence of a sequence of stability / instability intervals in the delay parameter space. Similar observations were made, for instance, in [8], where the robustness of stability of gyroscopic systems with respect to unmodeled time-delays was investigated, and in [28], in the framework of the stability analysis of delay models from machine tool vibrations applications. Also in the context of computing stability regions in the delay parameter space of systems with multiple delays the occurrence of multiple stability zones is well known, see, e.g. [9, 23, 26]. While most of these cited references are algorithmic in nature and focus on methods for computing stability regions in parameter spaces, the goal of this paper is to reveal the *mechanisms* which explain the observed phenomena (sequences of stability intervals in the delay parameter space, stability recovered for large delay values) and explain how delayed feedback works for such problems.

The stability analysis of oscillatory systems subject to delayed feedback is also a topic of recurrent interest in the physics literature. An important application area is the study of the dynamics of lasers characterized by delayed optical feedback, see, e.g., [12, 10] and the references therein. Furthermore, as we shall briefly discuss at the end, the methodology of the paper is also applicable to systems whose characteristic function takes the form

$$f(s) + \epsilon(g_1(s) + g_2(s)e^{-s\tau}). \tag{5}$$

This observation is of importance because the class of systems described by (5) includes oscillators subject to weak damping and weak delayed feedback, as well as oscillatory systems controlled with (Pyragas type) time-delayed feedback, see, e.g., [31, 30] and the references therein

The approach of the paper consists of studying the limiting behavior of the zeros of (1) as a function of the delay parameters on any a priori given compact delay intervals as $|\epsilon| \to 0$. As we shall see, for undamped oscillatory systems this allows us to make stability assertions for *any* delay value, thus also for delay values which are very large compared to the time-scale of the system. Furthermore, although the stability assertions for large delay values will typically involve small gain values, which reveals an underlying scaling property between the gain and the delay parameter, these parameters are not assumed to be coupled in the analysis, in contrast to [6], where time-scale separation techniques are used to analyze the solutions of harmonic oscillators subject to delayed perturbations with delay and gain parameters inversely proportional to each other. Finally, it is important to mention that, in studying the stability properties for large delay values, instead of fixing a delay interval that contains the targeted delays and analyzing the limiting behavior of the zeros of (1) as $|\epsilon| \rightarrow 0$, it is also possible to fix the gain and study the asymptotic behavior of the zeros as the delay parameter tends to infinity - the references [31, 30] rather lie in this direction. As both approaches lead to different insights in the problem, they complement each other.

The stabilization mechanism for a single oscillator subject to weak delayed feedback is related to the fact that the delay needs to provide an appropriate phase in the feedback loop. This is well known, see, e.g. [12] in the context of laser dynamics and [1] in the context of feedback control. For multiple oscillators, for instance, systems where the characteristic equation takes the form

$$\Pi_{i=1}^{\nu}(s^2+\omega_i^2)+\epsilon e^{-s\tau}=0,\ \omega_i>0,\ i=1,\ldots,\nu, \tag{6}$$

the revealed stabilization mechanism follows the same lines, and the stability condition eventually boils down to a synchronization requirement on the delay parameter, in the sense that the delay needs to provide an appropriate phase, compatible with *all* oscillatory modes.

Note that (6) can be interpreted as the characteristic equation of ν oscillators connected to each other in a ring configuration, with transmission delays in the coupling that summate to τ. In that sense the study of the zeros of (1) fits within the framework of the study of so-called delay induced amplitude death of oscillators, see e.g. [2, 3] and the references therein. In these references one investigates under which conditions the coupling of oscillators leads to a blocking of the periodic motion and results in a stable steady state solution.

To conclude the introduction, we mention mechanisms which are different from these discussed in the paper and where time-delays are beneficial in achieving stability. In [14, 21] time-delays appear in the control law that stem from a finite difference approximation of (missing) state or output derivatives. While the underlying mechanism is a high-gain mechanism in the sense that a good approximation of derivatives requires small delays and high gains (unless some scaling properties of the system can be exploited as in the multiple integrator case, see [21]) and the dynamic behavior of the closed-loop system is imposed by the control law, the mechanisms presented in this paper can be rather seen as dual: they are low-gain based and, therefore, rather *correct* the dynamic behavior of the open loop system. Control schemes based on finite spectrum assignment [17] or the Smith principle [27] also feature delays in the control laws, which are needed for the generation of predictions that compensate other delays in the control loop. We note that if the Pyragas controller [24] is used to stabilize an unstable periodic solution, then the delay parameter rather serves for the *selection* of the unstable periodic orbit that ones aims to stabilize (the delay must match the period of this solution) than for the stabilization itself. The matching between the delay and the period of the periodic orbit to be stabilized can however also been interpreted as a synchronization requirement.

The structure of the paper is as follows. In Section 2 the behavior of the zeros of the function (1) is studied. In Section 3 the results are applied to the stabilizability and stabilization of oscillatory systems using delayed output feedback. For multiple oscillators a relation between stabilization and a synchronization problem is established. Some examples are discussed in Section 4. The analysis of weakly damped or unstable oscillators is discussed in Section 5. The conclusions are presented in Section 6.

The following notation and definitions will be adopted: $\mathbb{C}$ ($\mathbb{C}_+$, $\mathbb{C}_-$) is the set of complex numbers (with strictly positive and strictly negative real parts), and $j = \sqrt{-1}$. For $s \in \mathbb{C}$, $\bar{s}$, $\Re(s)$ and $\Im(s)$ define the complex conjugate, the real part and the imaginary part of s. For $\Omega \subset \mathbb{C}$, $\partial\Omega$ denotes the boundary of Ω. $\mathbb{R}$ ($\mathbb{R}_+, \mathbb{R}_-$) denotes the set of real numbers (larger or equal to zero, smaller or equal to zero). $\mathbb{N}$ is the set of natural numbers and includes zero. $\mathbb{Z}$ the set of integers. For $s \in \mathbb{C}$ we denote by $s^{\frac{1}{m}}$ the number $c \in \mathbb{C}$ satisfying $c^m = s$ and $c = re^{j\phi}$ with $r \geq 0$ and $\phi \in (-\pi/m,\ \pi/m]$. A function $H:\ \mathbb{C} \to \mathbb{C}$ is called stable if all its zeros are in $\mathbb{C}_-$.

2 Behavior of zeros as a function of the delays

The following result characterizes the behavior of the zeros of (1) as a function of the delay parameter:

Proposition 2.1 *Let $\hat{s}$ be a zero of f with multiplicity $m \geq 1$ that is not a zero of g. Let $\Omega \subset \mathbb{C}$ be a compact set, which contains $\hat{s}$ but no other zeros of f, and such that $\partial\Omega$ is a closed, simple contour not containing $\hat{s}$. Then for all $\hat{\tau} > 0$ there exists a number $\hat{\epsilon} > 0$ such that the following holds:*

1. *$H(s;\ \tau, \epsilon)$ has exactly m zeros in Ω, for all $(\tau, \epsilon) \in [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}]$;*
2. *there are m functions $r_i:\ [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}] \to \Omega$,*
$$(\tau, \epsilon) \mapsto r_i(\tau, \epsilon), \quad i = 1, \ldots, m,$$
which satisfy $r_i(0,0) = \hat{s}$, $H(r_i(\tau,\epsilon);\ \tau, \epsilon) = 0$, $\forall (\tau, \epsilon) \in [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}]$, and which can be decomposed as:
$$r_i(\tau, \epsilon) = \hat{s} + |\epsilon|^{\frac{1}{m}} \mu_i(\tau, \epsilon),$$
where
$$\begin{array}{l} \lim_{|\epsilon| \to 0+} \max_{\tau \in [0,\ \hat{\tau}]} \left| \mu_i(\tau, \epsilon) - \right. \\ \left. \left(-\mathrm{sign}(\epsilon) \frac{m! g(\hat{s})}{\frac{d^m f}{ds^m}(\hat{s})} \right)^{\frac{1}{m}} e^{\frac{2\pi(i-1)j}{m}} e^{-\frac{\hat{s}}{m}\tau} \right| = 0,\ i = 1, \ldots, m. \end{array} \tag{7}$$

Proof. Given $\hat{\tau}$, there always exists a number $\hat{\epsilon} > 0$ such that for all $\tau \leq \hat{\tau}$ and $-\hat{\epsilon} \leq \epsilon \leq \hat{\epsilon}$, we have
$$|H(s;\ \tau, \epsilon) - f(s)| < |f(s)|,\ \forall s \in \partial\Omega. \tag{8}$$
By Rouché's Theorem (see Appendix A) it follows that H and f have the same number of zeros in Ω (this number is equal to m). This implies the existence of the functions $r_i,\ i = 1, \ldots, m$.

By making the substitution $s = \hat{s} + |\epsilon|^{\frac{1}{m}} z$ and scaling with ϵ, the function H becomes:
$$h(z;\ \tau, \epsilon) := f\left(\hat{s} + |\epsilon|^{\frac{1}{m}} z\right)/\epsilon + g\left(\hat{s} + |\epsilon|^{\frac{1}{m}} z\right) e^{-\hat{s}\tau} e^{-|\epsilon|^{\frac{1}{m}} z\tau}. \tag{9}$$

Define the function
$$\hat{h}(z;\ \tau, \epsilon) := \mathrm{sign}(\epsilon) \frac{1}{m!} \frac{d^m f}{ds^m}(\hat{s}) z^m + g(\hat{s}) e^{-\hat{s}\tau}, \tag{10}$$

which is obtained from (9) by truncating the Taylor expansions of the two terms around $z = 0$. The function (10) has m zeros, namely:

$$z_{0,i}(\tau, \epsilon) := \left(-\mathrm{sign}(\epsilon) \frac{m! g(\hat{s})}{\frac{d^m f}{ds^m}(\hat{s})} \right)^{\frac{1}{m}} e^{\frac{2\pi(i-1)j}{m}} e^{-\frac{\hat{s}}{m}\tau}, \quad i = 1, \ldots, m.$$

Next, we define $\tilde{\gamma} > 0$ and $\tilde{\epsilon} > 0$ such that

$$\tilde{\gamma} < \min_{\tau \in [0,\ \hat{\tau}]} |z_{0,i}(\tau, \epsilon)|, \tag{11}$$

$$\left\{ s \in \mathbb{C} :\ |s - \hat{s}| < 2|\tilde{\epsilon}|^{\frac{1}{m}} \max_{\tau \in [0,\ \hat{\tau}]} |z_{0,i}(\tau, \epsilon)| \right\} \subset \Omega, \tag{12}$$

$$\tilde{\epsilon} < \hat{\epsilon}. \tag{13}$$

Because $h(z;\ \tau, \epsilon)$ converges to $\hat{h}(z;\ \tau, \epsilon)$ as $|\epsilon| \to 0$, uniformly on compact subsets of $\mathbb{C}$ and compact delay intervals, we can conclude the following: for all $\gamma \in (0,\ \tilde{\gamma})$, there exists a number $\epsilon_\gamma \in (0,\ \tilde{\epsilon})$, such that for all $i \in \{1, \ldots, m\}$ the following estimate holds:

$$\left| h(z;\ \tau, \epsilon) - \hat{h}(z;\ \tau, \epsilon) \right| < |\hat{h}(z;\ \tau, \epsilon)|,$$
$$\forall \tau \leq \hat{\tau},\ \forall \epsilon \in [-\epsilon_\gamma,\ 0) \cup (0,\ \epsilon_\gamma],\ \forall z \in \mathbb{C} \text{ with } |z - z_{0,i}(\tau, \epsilon)| = \gamma. \tag{14}$$

An application of Rouché' s Theorem yields that the functions h and $\hat{h}$ have the same number of zeros in each of the disks

$$\{ z \in \mathbb{C} :\ |z - z_{0,i}(\tau, \epsilon)| < \gamma \},\ i = 1, \ldots, m. \tag{15}$$

Furthermore, because $\gamma < \tilde{\gamma}$ and $\tilde{\gamma}$ satisfies (11), the function $\hat{h}$, and thus also the function h, has a *unique* zero in each of the disks (15). Second, by the relations $|\epsilon| < \tilde{\epsilon}$ and the condition (12), the disks (15), when transformed to the s-plane, are subsets of Ω. Finally, by (13) the zeros of h in the disks (15) correspond to $\mu_i(\tau, \epsilon)$. We can therefore deduce the following from (14): for all $\gamma > 0$, there exists a number $\epsilon_\gamma \in (0,\ \tilde{\epsilon})$ such that

$$\max_{\tau \in [0,\ \hat{\tau}],\ 1 \leq i \leq m} |\mu_i(\tau, \epsilon) - z_0(\tau, \epsilon)| < \gamma,\ \ \forall \epsilon \in [-\epsilon_\gamma,\ 0) \cup (0,\ \epsilon_\gamma].$$

This statement is equivalent with the assertion (7). □

For the case of a simple zero ($m = 1$) Proposition 2.1 simplifies to:

Corollary 2.2 *Let $\hat{s}$ be a zero of f with multiplicity one that is not a zero of g. Let $\Omega \subset \mathbb{C}$ be a compact set, which contains $\hat{s}$ but no other zeros of f, and such that $\partial\Omega$ is a closed, simple contour not containing $\hat{s}$. Then for all $\hat{\tau} > 0$ there exists a number $\hat{\epsilon} > 0$ such that $H(s;\ \tau, \epsilon)$ has exactly one zero in Ω, for all $(\tau, \epsilon) \in [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}]$. Furthermore, there is a (unique) function $r :\ [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}] \to \Omega$, $(\tau, \epsilon) \mapsto r(\tau, \epsilon)$, which satisfies $r(0, 0) = \hat{s}$,*

$$H(r(\tau, \epsilon);\ \tau, \epsilon) = 0,\ \ \forall (\tau, \epsilon) \in [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}],$$

and which can be decomposed as:

$$r(\tau, \epsilon) = \hat{s} + \epsilon\ \mu(\tau, \epsilon),$$

where

$$\lim_{|\epsilon| \to 0+} \max_{\tau \in [0,\ \hat{\tau}]} \left| \mu(\tau, \epsilon) + \frac{g(\hat{s})}{f'(\hat{s})} e^{-\hat{s}\tau} \right| = 0. \tag{16}$$

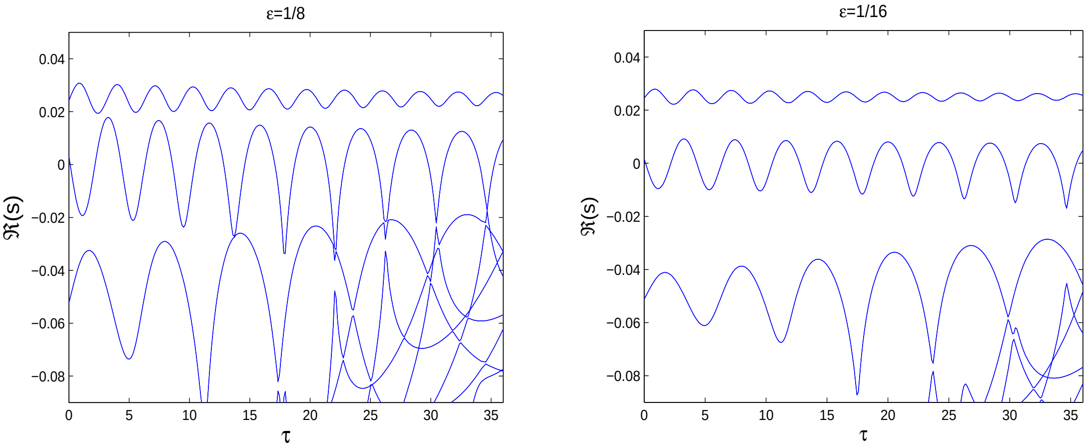

Figure 1: The real parts of the rightmost zeros of (18) as a function of the delay parameter τ, for small values of the gain parameter ϵ.

We briefly discuss the above results, starting with the special case treated in the Corollary 2.2. Expression (16) implies that for small values of the gain parameter ϵ, an isolated zero behaves as the function

$$\tau \mapsto \hat{s} - \epsilon \frac{g(\hat{s})}{f'(\hat{s})} e^{-\hat{s}\tau}. \tag{17}$$

Consequently, if $\Im(\hat{s}) > 0$, it has an oscillatory behavior as a function of the delay parameter. If, furthermore, $\hat{s} \in \mathbb{C}_-$ $(\mathbb{C}_+)$, then the zero behaves like an unstable (stable) *spiral*. This is illustrated in Figure 1 for the quasi-polynomial

$$H(s;\ \tau, \epsilon) = \prod_{i=1}^{6} (s - s_i) + \epsilon e^{-s\tau}, \tag{18}$$

where

$$s_{1,2} = -\frac{1}{20} \pm j,\ \ s_{3,4} = \pm\frac{3}{2}j,\ \ s_{5,6} = \frac{1}{40} \pm 2j,$$

$\tau \in [0,\ 36]$ and $\epsilon = 1/8$ (left), respectively $\epsilon = 1/16$ (right). The zeros of (18) were computed using the package DDE-BIFTOOL [7]. Notice from (16) that $\mu(\tau, \epsilon)$ is the offset of the zero, i.e. $r(\tau, \epsilon) - \hat{s}$, *scaled* by the gain parameter ϵ. This explains why the 'amplitudes' of the oscillations in Figure 1 are smaller for $\epsilon = 1/16$ than for $\epsilon = 1/8$.

In the case where f has a zero with multiplicity $m > 1$, covered by Proposition 2.1, the m corresponding zeros of H behave for fixed small values of the gain parameter as the equally 'shifted' spiral curves,

$$\tau \mapsto \hat{s} + \left(-\epsilon \frac{m! g(\hat{s})}{\frac{d^m f}{ds^m}(\hat{s})} \right)^{\frac{1}{m}} e^{-\frac{\hat{s}}{m}\tau} e^{j\frac{2\pi(i-1)j}{m}},\ \ i = 1, \ldots, m.$$

To illustrate this we show in Figure 2 the real parts of the rightmost zeros of

$$H(s;\ \tau, \epsilon) = (s^2 + 1)^3 + \epsilon e^{-s\tau} \tag{19}$$

as a function of τ for $\epsilon = 1/640$. The (initial) delay-shift of 2π between the functions (corresponding to a *phase* shift of $2\pi/3$) is a consequence of the presence of a zero of $f(s) = h(s;\ \tau, 0)$ with multiplicity three.

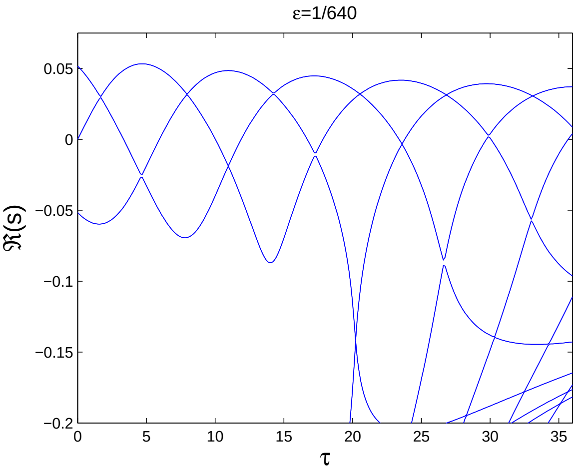

Figure 2: The real parts of the rightmost zeros of (19) as a function of the delay parameter τ, for $\epsilon = 1/640$.

3 Stabilization of oscillatory systems

In this section we assume that the rightmost zeros of f are on *the imaginary axis.* We further assume that

$$f(\bar{s}) = \overline{f(s)},\ g(\bar{s}) = \overline{g(s)},\ \forall s \in \mathbb{C}.$$

As a consequence, the zeros of H appear in complex conjugate pairs.

3.1 Simple zeros

If all zeros of f on the imaginary axis are simple, then the corresponding functions of the form (17) have a *sinusoidal* real part, exhibiting different frequencies. As a consequence, the asymptotic stability of H for small values of ϵ is related to having *an appropriate phase* of these sinusoidal functions, which depends of the *delay parameter only.* This relation between the stability of H and a phase '*synchronization*' problem is clarified in the following proposition:

Proposition 3.1 *Assume that there is a constant $\gamma > 0$ such that*

$$\lim_{R\to\infty} \sup \left\{ \left| \frac{g(s)}{f(s)} \right| :\ \Re(s) \geq -\gamma,\ |s| \geq R \right\} = 0. \tag{20}$$

Assume further that all zeros of f are in the closed left half plane. Denote by $j\omega_i$, $i = 1, \ldots \nu$, the zeros of f on the positive imaginary axis, which all have multiplicity one.
If the delay parameter τ is such that for all $i = 1, \ldots, \nu$:

$$\Re\left(\frac{g(j\omega_i)}{f'(j\omega_i)} e^{-j\omega_i\tau} \right) > 0\ (< 0), \tag{21}$$

then all zeros of $H(s;\ \tau, \epsilon)$ are in the open left half plane for sufficiently small $\epsilon > 0$ ($\epsilon < 0$).

Remark 3.2 *The assumption (20) is technical and serves to exclude the situation where increasing* $|\epsilon|$ *from zero leads to the introduction of additional zero's in the right half plane that come from infinity. It is fulfilled in most applications of interest. For instance, it is satisfied if* f *and* g *are polynomials satisfying* $\deg(g) < \deg(f)$*.*

Proof. Let τ_0 be a delay value for which

$$\Re\left(\frac{g(j\omega_i)}{f'(j\omega_i)}e^{-j\omega_i\tau_0}\right) > 0,\ i = 1, \ldots, \nu. \tag{22}$$

Consider the functions $v_i:\ \mathbb{R}_+ \rightarrow \mathbb{C}$,

$$\tau \mapsto v_i(\tau;\ \epsilon) = j\omega_i - \epsilon\frac{g(j\omega_i)}{f'(j\omega_i)}e^{-j\omega_i\tau},\quad i = 1, \ldots, \nu. \tag{23}$$

The condition (22) assures that

$$\Re(v_i(\tau_0;\ \epsilon)) < 0,\ i = 1, \ldots, \nu.$$

By Corollary 2.2 this implies that $H(s;\ \tau_0, \epsilon)$ has ν zeros, $s_i(\epsilon)$, in $\mathbb{C}_-$ for small positive values of ϵ, which satisfy

$$\lim_{\epsilon\rightarrow 0+}\frac{|v_i(\tau_0;\ \epsilon) - s_i(\epsilon)|}{\epsilon} = 0,\ i = 1, \ldots, \nu. \tag{24}$$

Next, let N be the number of zeros of f on the imaginary axis ($N = 2\nu$ if $\omega_i \neq 0,\ i = 1, \ldots, \nu$, and $N = 2\nu - 1$ otherwise). Choose $\gamma > 0$ such that (20) holds, f has no zeros with real part equal to $-\gamma$ and only N zeros in with real part larger than $-\gamma$. From (20) there exists a number $\hat{R} > 0$ such for all $s \in \mathbb{C}$ satisfying $\Re(s) \geq -\gamma$ and $|s| \geq \hat{R}$, we have

$$\left|\frac{g(s)}{f(s)}e^{-s\tau_0}\right| < 1,$$

which implies that $H(s;\ \tau_0, \epsilon) \neq 0$ for $\epsilon \in [-1,\ 1]$. It follows that for all $\epsilon \in [-1,\ 1]$, the zeros of $H(s;\ \tau_0, \epsilon)$ in the half plane $\{s \in \mathbb{C}:\ \Re(s) \geq -\gamma\}$ are confined to the set

$$\Omega := \{s \in \mathbb{C}:\ \Re(s) \geq -\gamma,\ |s| \leq R\}.$$

Since $H(s;\ \tau_0, \epsilon) \rightarrow f(s)$ as $|\epsilon| \rightarrow 0$, uniformly on Ω, it follows from Rouché's Theorem (Appendix A) that for sufficiently small values of ϵ the number of zeros of f and H in Ω are both equal to N. When combining this fact with the continuity of the individual zeros of H with respect to parameter ϵ and the expressions (24), it follows that $H(s;\ \tau_0, \epsilon)$ is stable for small positive values of ϵ.

The proof is similar for the case where $\Re\left(\frac{g(j\omega_i)}{f'(j\omega_i)}e^{-j\omega_i\tau_0}\right) < 0,\ i = 1, \ldots, \nu.$ □

Remark 3.3 *Proposition 3.1 generalizes Proposition 5.3 of [13], where an alternative proof is presented, based on the sensitivity of the zeros of* H *with respect to the parameter* ϵ *for a fixed value of* τ*. More precisely, define a function* $s_i(\tau, \epsilon)$ *satisfying* $s_i(0, 0) = j\omega_i$ *and* $H(s_i(\tau, \epsilon);\ \tau, \epsilon) = 0$*. Then we get:*

$$\left.\frac{\partial s_i(\tau, \epsilon)}{\partial\epsilon}\right|_{\epsilon=0} = -\frac{g(j\omega_i)e^{-j\omega_i\tau}}{f'(j\omega_i)},\ i = 1, \ldots, \nu. \tag{25}$$

Expression (21) is equivalent to requiring

$$\left.\frac{\partial\Re(s_i(\tau, \epsilon))}{\partial\epsilon}\right|_{\epsilon=0} < 0,\ i = 1, \ldots, \nu.$$

The next result addresses a case where the condition (21) can always be met by an appropriate choice of the delay parameter:

Corollary 3.4 *Assume that all zeros of f are in the closed left half plane. Denote by $j\omega_i,\ i = 1, \ldots \nu$, the zeros of f on the positive imaginary axis, which all have multiplicity one. Assume further that (20) holds. If the nonzero frequencies ω_i are rationally independent*[1]*, then there always exist values of ϵ and τ such that the zeros of $H(s;\ \tau, \epsilon)$ are in $\mathbb{C}_-$.*

Proof. Because the nonzero frequencies ω_i are rationally independent, Kronecker's theorem [11, Theorem 444] can be applied: given arbitrary $\theta_i \in [0,\ 2\pi]$, $i = 1, \ldots, m$, there always exists a sequence $\{\tau_n\}_{n\geq 1}$ such that

$$\left| e^{j\theta_i} - e^{-j\omega_i \tau_n} \right| < \frac{1}{n}, \ \ \forall i \in \{1, \ldots, m\}, \omega_i \neq 0.$$

As a consequence, the delay can always be chosen such that the condition (21) is satisfied. □

If the frequencies $\omega_i,\ i = 1, \ldots, \nu$, are commensurate (multiples of the same number), then the condition (21) can be easily checked because the sinusoidal functions

$$\tau \mapsto \Re\left(\frac{g(j\omega_i)}{f'(j\omega_i)} e^{-j\omega_i \tau} \right),\ i = 1, \ldots, \nu,$$

need to be evaluated only on a finite interval of length $2\pi/(\min_i \omega_i)$. With the following example we illustrate that situations might occur where the condition (21) is violated, whatever the value of the delay.

Example 3.5 *Let*

$$f(s) = \Pi_{i=1}^{5}(s^2 + \omega_i^2), \quad g(s) \equiv 1,$$

where $\omega_1 = 1,\ \omega_2 = 3,\ \omega_3 = 4,\ \omega_4 = 5,\ \omega_5 = 6$. In order to check condition (21), we compute

$$\left\{ \Re\left(\frac{g(j\omega_i)}{f'(j\omega_i)} e^{-j\omega_i \tau} \right) \right\}_{i=1}^{5} = \\ \left\{ \frac{-1}{201600} \sin(\tau),\ \frac{1}{145152} \sin(3\tau),\ \frac{-1}{151200} \sin(4\tau),\ \frac{1}{380160} \sin(5\tau),\ \frac{-1}{2494800} \sin(6\tau) \right\}. \tag{26}$$

Because there is no value of $\tau \in [0,\ 2\pi]$ such that all elements of (26) are either strictly positive or strictly negative, the condition (21) cannot be satisfied.

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[1] The real numbers $(r_1, r_2, \ldots, r_m)$ are *rationally independent* if and only if $\sum_{k=1}^{m} n_k r_k = 0, \quad n_k \in \mathbb{Z}$, implies $n_k = 0, \quad k = 1, \ldots, m$. For example, two numbers are rationally independent if and only if their ratio is an irrational number.

3.2 Multiple zeros

According to Proposition 2.1 the zeros of $H(s;\ \tau,\epsilon)$ that correspond to a zero $j\omega, \omega \geq 0$, of f with multiplicity $m > 1$ behave for small values of ϵ as the functions

$$\tau \mapsto w_i(\tau) := j\omega + \left(-\epsilon \frac{m!g(j\omega)}{\frac{d^m f}{ds^m}(j\omega)}\right)^{\frac{1}{m}} e^{-\frac{j\omega}{m}\tau} e^{\frac{2\pi(i-1)j}{m}},\ i = 1, \dots, m.$$

Consequently, if $m \geq 2$, then there does not exist a delay value such that

$$\Re(w_i(\tau)) < 0, \quad i = 1, \dots, m.$$

Moreover, if $m > 2$, then for every value of τ, there exists an index $k \in \{1, \dots, m\}$ such that $\Re(w_k(\tau)) > 0$. It follows from Proposition 2.1 that there does *not* exist a sequence of stabilizing pairs $\{(\tau_n, \epsilon_n)\}_{n \geq 0}$ with $\lim_{n\to\infty} \tau_n < \infty$ and $\lim_{n\to\infty} \epsilon_n \to 0$. If $m = 2$, there are values of τ for which

$$\Re(w_i(\tau)) = 0,\ i = 1, 2. \tag{27}$$

In this case a higher order analysis of the behavior of the zeros is needed to conclude about stability for small values of ϵ, excepting in special situations, as we now illustrate.

Using a different approach the following result can be derived, which corresponds to a reformulation of Corollary III.4 of [13] to the problem discussed in this paper:

Proposition 3.6 *If f is a polynomial having a zero in the closed right half plane with multiplicity m and if g is a polynomial satisfying* $\deg(g) < m-1$, *then there does not exist a pair (ϵ, τ) such that $H(s;\ \tau, \epsilon) = f(s) + \epsilon g(s)e^{-s\tau}$ has all zeros in $\mathbb{C}_-$.*

Proof. The proof is by contradiction. Assume that for some pair (ϵ, τ) the zeros of $H(s;\ \tau, \epsilon)$, which are equal to the zeros of

$$J(s;\ \tau, \epsilon) := f(s)e^{s\tau} + \epsilon g(s), \tag{28}$$

are in $\mathbb{C}_-$. Lemma II.1 of [13], an extension of Lukas' theorem to quasi-polynomials, states that if a function of the form (28) has its zeros confined to $\mathbb{C}_-$, than all its derivatives have their zeros confined to $\mathbb{C}_-$. Consequently, also the zeros of

$$\frac{d^{m-1}J(s;\ \tau, \epsilon)}{ds^{m-1}} = \frac{d^{m-1}f(s)e^{s\tau}}{ds^{m-1}} = e^{s\tau}\left(\left(\frac{d}{ds} + \tau\right)^{m-1} f(s)\right)$$

must be in $\mathbb{C}_-$. But this contradicts the fact that

$$\frac{d^{m-1}J(s_0;\ \tau, \epsilon)}{ds^{m-1}} = 0,$$

where s_0 is a zero of f in the closed right half plane with multiplicity m. □

Under the assumptions taken on f and g, Proposition 3.6 strengthens the results that can be obtained from Proposition 2.1 in two ways:

1. the statement of Proposition 3.6 is global in ϵ, in the sense that it does not restrict to assertions about small values of this parameter;
2. if $m = 2$, and $\deg(g(s)) = 0$, then it shows that stability is not possible. This implies that for a fixed value of τ satisfying condition (27), the situation where the locus of zeros of H as a function ϵ is tangential to the imaginary axis at $\epsilon = 0$ and bend backwards towards the left half plane can be excluded.

4 Examples

We illustrate the results obtained in the paper with two examples. With the first one (single oscillator controlled with delayed feedback) we further illustrate the results obtained in Section 2. With the second example (a gyroscopic system) we emphasize the phase-synchronization point of view towards stabilization, described in Section 3.

Both examples concern linear time-delay systems of retarded type, where the characteristic function takes the form (1), with f and g polynomials satisfying $\deg(f(s)) > \deg(g(s))$. For spectral properties of linear time-delay systems of retarded type and their relation with stability properties we refer to, e.g., [4, 29, 28] and the references therein. Most important for the remainder of the paper are the following: a steady state solution is asymptotically stable if and only if all zeros of the corresponding characteristic function are in $\mathbb{C}_-$, and, although the characteristic function generically has an infinite number of zeros, the number of zeros in *any* right half plane, i.e. $\{s \in \mathbb{C} : \ \Re(s) \geq r\}, \ r \in \mathbb{R}$, is finite.

4.1 Oscillator with delayed output feedback

We analyze the stability of the feedback interconnection of the oscillator (4) with the control law $u(t) = -\epsilon y(t-\tau)$ (or $U(s) = -\epsilon e^{-s\tau}\ Y(s)$ in the frequency domain). The characteristic function of the closed-loop system takes the form $H(s;\ \tau, \epsilon) := f(s) + \epsilon g(s) e^{-s\tau}$, where

$$f(s) = s^2 + \Omega^2, \ g(s) = 1.$$

The following result from [22] characterizes the stability regions in the (ϵ, τ) parameter space (see also [1]):

Proposition 4.1 *Let* $f(s) = s^2 + \Omega^2$ *and* $g(s) \equiv 1$*. The quasi-polynomial* $H(s;\ \tau, \epsilon) = f(s) + \epsilon g(s) e^{-s\tau}$ *is stable if and only if*

$$\epsilon \in (-\Omega^2,\ 0) \cup \left(0,\ \tfrac{3}{5}\Omega^2\right), \quad \tau \in \left(\tau_-^{(1)},\ \tau_+^{(1)}\right) \cup \ldots \cup \left(\tau_-^{(l)},\ \tau_+^{(l)}\right),$$

where

$$\tau_-^{(k)} = \pi \frac{\left(\frac{1+\mathrm{sign}(\epsilon)}{2} + 2k\right)}{\sqrt{\Omega^2 - |\epsilon|}}, \ \tau_+^{(k)} = \pi \frac{\left(\frac{3+\mathrm{sign}(\epsilon)}{2} + 2k\right)}{\sqrt{\Omega^2 + |\epsilon|}}, \quad k \in \mathbb{N},$$

and

$$l = \max\left\{k \in \mathbb{N} :\ \tau_-^{(k)} < \tau_+^{(k)}\right\}.$$

Remark 4.2 *The condition on the gain* ϵ *assures that* $l \geq 1$*.*

The number of stability intervals in the delay parameter space, l, is always finite but can be made *arbitrarily large* by letting $|\epsilon| \to 0$. This fact, as well as the sequence of stability / instability intervals, are explained by the oscillatory behavior of the rightmost characteristic roots of the closed loop system as a function of the delay parameter, described in Corollary 2.2, and, in particular, by their relation with the function

$$\tau \mapsto j\Omega - \epsilon \frac{g(j\Omega)}{f'(j\Omega)} e^{-j\Omega\tau} = j\Omega + \epsilon \frac{j}{2\Omega} e^{-j\Omega\tau}.$$

4.2 Gyroscopic system

In [8] one investigates the effect of time-delays on the stability of a mechanical system with gyroscopic forces, modeled by systems of differential equations of the form

$$My''(t) + Ty'(t-r) + Ky(t) = 0. \tag{29}$$

Here, y and y' represent displacements and velocities. $M > 0$ and K are real symmetric matrices which are usually referred to as the mass matrix and the stiffness matrix, and correspond to inertial and potential forces, respectively. The matrix T is skew-symmetric and the corresponding term models the gyroscopic forces. The parameter $r \geq 0$ represents a time-delay in their application. In Section 5 of [8] a procedure for the determination of the stability regions in the delay parameter space is presented for the second-order case, where (29) can be brought in the form:

$$y''(t) + \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} y'(t-r) + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} y(t) = 0, \tag{30}$$

with κ, ω_1 and ω_2 real constants.

The characteristic function of the system (30) is given by

$$H(s;\ \tau, \epsilon) = f(s) + \epsilon g(s) := (s^2 + \omega_1^2)(s^2 + \omega_2^2) + \epsilon s^2 e^{-s\tau}, \tag{31}$$

where $\epsilon = \kappa^2$ and $\tau = 2r$. In the sequel we discuss the stability regions of (30) in the (τ, ϵ) parameter space, where we do not necessarily restrict ourselves to positive ϵ as this assumption is not needed in the general theory developed in the paper. However, from the application point of view only the obtained results for $\epsilon \geq 0$ are relevant.

For $\omega_1 = 2$ and $\omega_2 = 4$ the functions $\upsilon_i :\ \mathbb{R}_+ \to \mathbb{R}$,

$$\tau \mapsto \upsilon_i(\tau) = -\Re\left(\frac{g(j\omega_i)}{f'(j\omega_i)} e^{-j\omega_i\tau}\right),\ i = 1, 2, \tag{32}$$

are depicted in Figure 3. Because $\deg(f(s)) > \deg(g(s)$ the assumption (20) of Proposition 3.1 is satisfied. According to this proposition asymptotic stability is achieved for small positive values of ϵ if $\upsilon_1(\tau) < 0$ and $\upsilon_2(\tau) < 0$, or, equivalently,

$$\tau \in \bigcup \left\{ \left(\frac{\pi}{4} + k\pi,\ \frac{\pi}{2} + k\pi\right) :\ k \in \mathbb{N} \right\}. \tag{33}$$

Similarly, asymptotic stability is achieved for small negative values of ϵ if $\upsilon_1(\tau_1) > 0$ and $\upsilon_2(\tau_2) > 0$, or

$$\tau \in \bigcup \left\{ \left(\frac{\pi}{2} + k\pi,\ \frac{3\pi}{4} + k\pi\right) :\ k \in \mathbb{N} \right\}. \tag{34}$$

The intervals (33) and (34) are indicated in Figure 3.

To illustrate the relation between the functions (32) and the behavior of the zeros of (31), described by Corollary 2.2, we have used the package DDE-BIFTOOL to compute the rightmost zeros of (31) as a function of the delay parameter τ, for $\omega_1 = 2$, $\omega_2 = 4$ and both $\epsilon = 1$, which corresponds to Example 5.1 of [8], and $\epsilon = 1/4$. The results are displayed in Figure 4. Notice the correspondence with the functions displayed in Figure 3. For a fixed delay value, namely $\tau = 4\pi/3$, the rightmost zeros of (31) are shown in Figure 5 as a function of the gain parameter ϵ. Clearly stability is obtained for small positive values of ϵ.

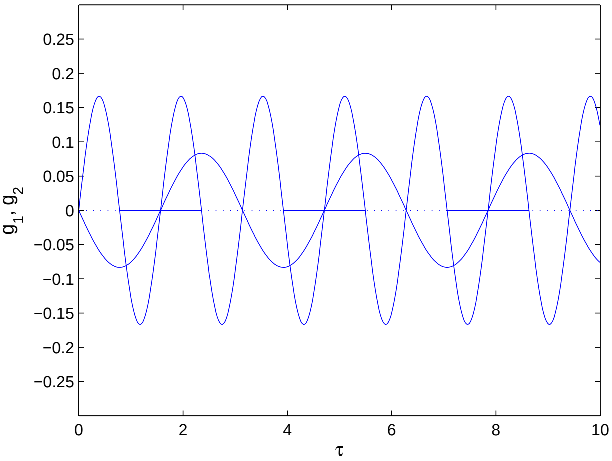

Figure 3: The sinusoidal functions (32). The intervals (33)-(34) are also indicated.

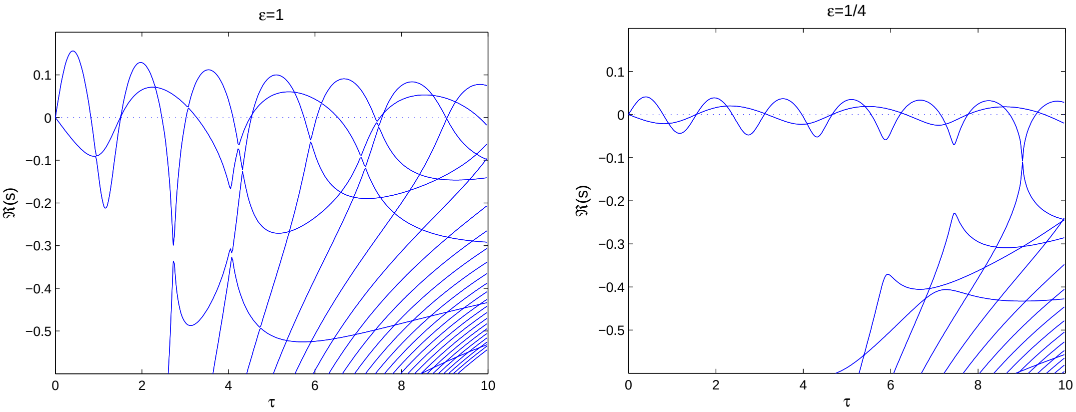

Figure 4: The real parts of the rightmost zeros of (31) as a function of τ, for $\omega_1 = 2$ and $\omega_2 = 4$. The left frame corresponds to $\epsilon = 1$, the right frame to $\epsilon = 1/4$.

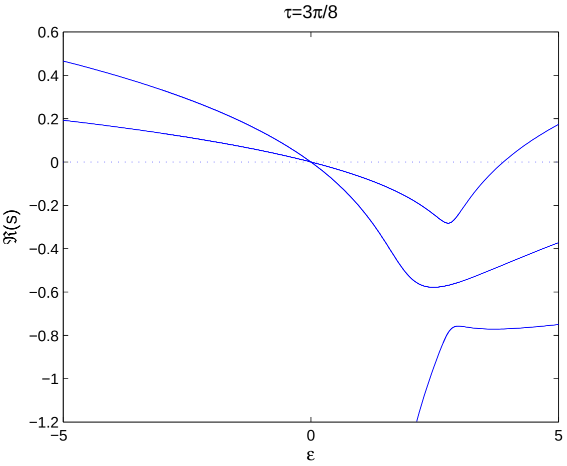

Figure 5: The real parts of the rightmost zeros of (31) as a function of ϵ for $\tau = 3\pi/8$, $\omega_1 = 2$ and $\omega_2 = 4$.

We note that the determination of a precise threshold on ϵ for guaranteeing stability (the assertion of Proposition 3.1 holds for 'sufficiently small' ϵ) and a quantification of the convergence rate behind the limit in Corollary 2.2 require a complete characterization or computation of the stability regions in the delay parameter space. For the quasi-polynomial (31) this can be done analytically, based on the following results, whose proof can be found in Appendix B:

Proposition 4.3 *Consider the function H, defined in (31), where $\omega_1 < \omega_2$. Assume that*

$$\epsilon \in \left(-(\omega_2 - \omega_1)^2,\ 0\right) \cup \left(0,\ (\omega_2 - \omega_1)^2\right). \tag{35}$$

Define the following numbers:

$$\tilde{\omega}_1 = \omega_1 \sqrt{1 + \frac{\epsilon}{2\omega_1^2} + \frac{\omega_2^2 - \omega_1^2}{2\omega_1^2}\left(1 - \sqrt{1 + \frac{\epsilon^2 + 2\epsilon(\omega_1^2 + \omega_2^2)}{(\omega_1^2 - \omega_2^2)^2}}\right)},$$

$$\tilde{\omega}_2 = \omega_1 \sqrt{1 - \frac{\epsilon}{2\omega_1^2} + \frac{\omega_2^2 - \omega_1^2}{2\omega_1^2}\left(1 - \sqrt{1 + \frac{\epsilon^2 - 2\epsilon(\omega_1^2 + \omega_2^2)}{(\omega_1^2 - \omega_2^2)^2}}\right)},$$

$$\tilde{\omega}_3 = \omega_2 \sqrt{1 - \frac{\epsilon}{\omega_2^2} - \frac{\omega_2^2 - \omega_1^2}{2\omega_2^2}\left(1 - \sqrt{1 + \frac{\epsilon^2 - 2\epsilon(\omega_1^2 + \omega_2^2)}{(\omega_1^2 - \omega_2^2)^2}}\right)},$$

$$\tilde{\omega}_4 = \omega_2 \sqrt{1 + \frac{\epsilon}{2\omega_2^2} - \frac{\omega_2^2 - \omega_1^2}{2\omega_2^2}\left(1 - \sqrt{1 + \frac{\epsilon^2 + 2\epsilon(\omega_1^2 + \omega_2^2)}{(\omega_1^2 - \omega_2^2)^2}}\right)}.$$

The function (31) has a zero $j\omega$, $\omega > 0$, for some delay value τ if and only if

$$\omega \in \{\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4\}. \tag{36}$$

Furthermore, for each ω satisfying (36) the corresponding delay values are given by

$$\begin{array}{lll} \mathcal{T}_{\tilde{\omega}_1} = \left\{\frac{2\pi l}{\tilde{\omega}_1},\ l \in \mathbb{N}\right\}, & & \omega = \tilde{\omega}_1, \\ \mathcal{T}_{\tilde{\omega}_2} = \left\{\frac{(2l+1)\pi}{\tilde{\omega}_2},\ l \in \mathbb{N}\right\} & & \omega = \tilde{\omega}_2, \\ \mathcal{T}_{\tilde{\omega}_3} = \left\{\frac{(2l+1)\pi}{\tilde{\omega}_3},\ l \in \mathbb{N}\right\} & & \omega = \tilde{\omega}_3, \\ \mathcal{T}_{\tilde{\omega}_4} = \left\{\frac{2\pi l}{\tilde{\omega}_4},\ l \in \mathbb{N}\right\} & & \omega = \tilde{\omega}_4. \end{array} \tag{37}$$

If $\epsilon > 0$, then the crossing direction of a zero $j\omega$, $\omega > 0$, as the delay is increased, is towards instability (stability) if $\omega \in \{\tilde{\omega}_2, \tilde{\omega}_4\}$ ($\omega \in \{\tilde{\omega}_1, \tilde{\omega}_3\}$). If $\epsilon < 0$, then the crossing direction is towards instability (stability) if $\omega \in \{\tilde{\omega}_1, \tilde{\omega}_3\}$ ($\omega \in \{\tilde{\omega}_2, \tilde{\omega}_4\}$).

Proposition 4.4 *Consider the function H, defined in (31), where $\omega_1 < \omega_2$. If $|\epsilon| \geq (\omega_2 - \omega_1)^2$, then there does not exist a value of τ such that H is stable.*

From Proposition 4.4 we can restrict ourselves to the case where (35) is satisfied in searching for stability intervals. For a given value of ϵ, satisfying this condition, the set

$$\mathcal{T} = \bigcup_{i \in \{1,\ldots,4\}} \mathcal{T}_{\tilde{\omega}_i}, \tag{38}$$

with $\mathcal{T}_{\hat{\omega}_i}$ defined in Proposition 4.3, leads to a partition of the delay space ($\mathbb{R}_+$) into intervals in which the number of zeros of H in the open right half plane is constant. The characterization of the stability regions in the delay parameter space is complete when taking into account the number of zeros of $H(s;\ 0, \epsilon)$ in the open right half plane (zero) and the assertion on the crossing direction, in the sense of [5].

Let us apply Proposition 4.3 to the case where $\omega_1 = 2$ and $\omega_2 = 4$. For $\epsilon = 1$ a necessary and sufficient stability condition is given by

$$\tau \in (0.8215,\ 1.5017) \cup (4.1077,\ 4.5051) \cup (7.3938,\ 7.5085),$$

and for $\epsilon = 1/4$ by

$$\begin{aligned} \tau \in (0.7938,\ 1.5542) \cup (3.9689,\ 4.6627) \cup\ & (7.1439,\ 7.7711) \cup (10.3190,\ 10.8796) \\ & \cup (13.4941,\ 13.9881) \cup (16.6692,\ 17.0965) \cup (19.0436,\ 20.2050). \end{aligned}$$

This is in accordance with the results shown in Figure 4 and the results presented in [8]. If $|\epsilon| \to 0$, then $\tilde{\omega}_1$ and $\tilde{\omega}_2$ converge to ω_1 and $\tilde{\omega}_3$ and $\tilde{\omega}_4$ converge to ω_2, while the corresponding partition (38) of the delay parameter space leads to the stability regions (33)-(34).

Finally, we consider the quasi-polynomial (31), under the assumption $\omega_1 = \omega_2$. According to Proposition 3.6 H is unstable for all values of ϵ and τ. Figure 6 shows the rightmost zeros as a function of ϵ for $\omega_1 = \omega_2 = 1$ and $\tau = 4\pi/3$. The high sensitivity of the zeros of H with respect to changes of ϵ near $\epsilon = 0$ is caused by the zero of f with multiplicity two. Since

$$\frac{\partial H}{\partial \epsilon}(s;\ \tau, \epsilon) \neq 0$$

for all $\tau \geq 0$, the graph of the rightmost zeros as a function of ϵ near $\epsilon = 0$ can be decomposed into two smooth curves, one for $\epsilon \geq 0$ and one for $\epsilon \leq 0$, which both exhibit a *turning point* at $\epsilon = 0$ (see, e.g. [25] for an introduction to bifurcation theory).

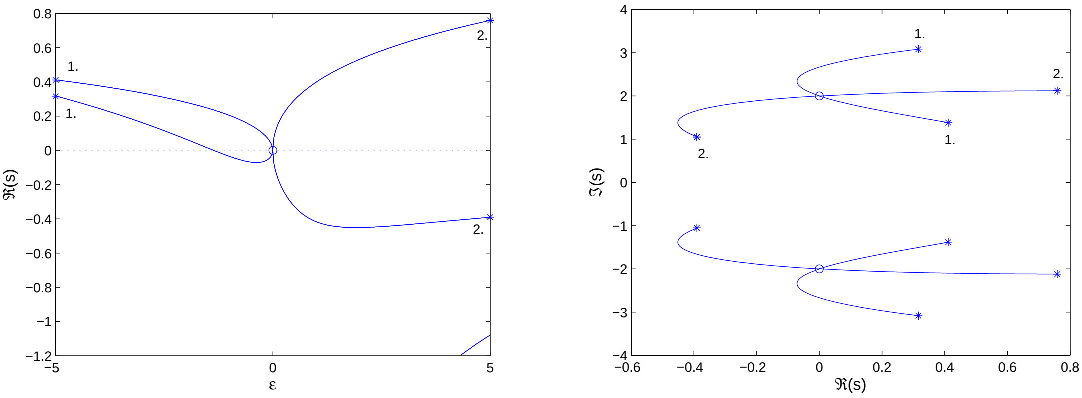

Figure 6: The real parts of the rightmost zeros of (31) as a function of ϵ (left) and the corresponding locus in the complex plane (right); $\omega_1 = \omega_2 = 1,\ \tau = 4\pi/3$.

5 Weakly unstable and weakly damped oscillators

With some examples we illustrate that the methodology of the paper can also be applied to systems which are unstable or weakly damped oscillators for $\epsilon = 0$, and that the stabilization mechanisms are similar. In the discussion we highlight similarities and differences with the case of undamped oscillators, mainly treated in Sections 3-4.

In contrast to damped and undamped oscillators, it is for unstable oscillators not possible to make assertions about the existence of stabilizing pairs (ϵ, τ) or intervals from the asymptotic behavior of the zeros as a function of the delay parameter (for $|\epsilon| \to 0$) *only*. In particular, the choice of the gain parameter may involve a trade off, as we now illustrate:

Example 5.1 *Consider the function*

$$H(\lambda;\ \tau, \epsilon) = ((s-a)^2 + \Omega^2) + \epsilon e^{-s\tau}, \tag{39}$$

where $a > 0$. According to Corollary 2.2 this function has a zero, which behaves for small values of ϵ as the function

$$\tau \mapsto a + j\Omega - \epsilon \frac{1}{2j\Omega} e^{-a\tau} e^{-j\Omega\tau}. \tag{40}$$

For $a = 1/50$ and $\Omega = 1$ we display in Figure 7 the real parts of the rightmost zeros of the function (39) as a function of τ, for $\epsilon = 1$ (upper left frame), $\epsilon = 1/20$ (upper right frame) and $\epsilon = 1/50$ (lower frame). Clearly, $\epsilon = 1$ is too large for having a good correspondence between the behavior of the rightmost zeros and the function (40). For $\epsilon = 1/20$ the correspondence is already better and explains the sequence of stability / instability intervals in the delay parameter space. For $\epsilon = 1/50$, the correspondence on the delay interval $[0, 45]$ is very good, yet the value of ϵ is too small to compensate the instability.

The situations is however different when the damping is also proportional to the gain parameter, as for the function

$$h(s;\ \tau, \epsilon) = s^2 + \Omega^2 + \epsilon(as + e^{-s\tau}), \tag{41}$$

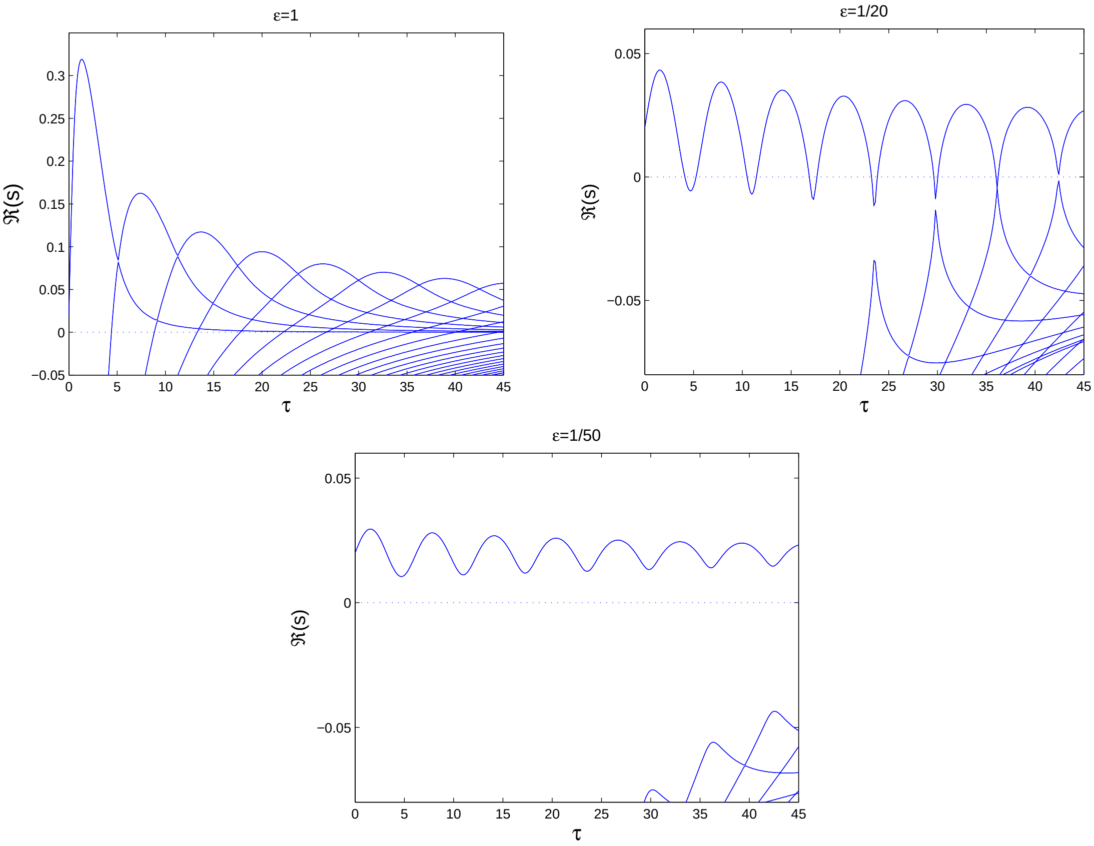

Figure 7: The real parts of the rightmost zeros of (39) as a function of τ for $a = 1/50$ and $\Omega = 1$. The top left frame corresponds to $\epsilon = 1$, the top right frame to $\epsilon = 1/20$ and the bottom frame to $\epsilon = 1/50$.

where $a \in \mathbb{R}$. Although this function is not of the form (1) the results of the paper can easily be adapted to such type of problems. For instance, one can derive the following result, whose proof follows completely the lines of the proof of Proposition 2.1:

Proposition 5.2 *Let* $\Omega = \{s \in \mathbb{C} : \ |s - j\Omega| < \Omega/2\}$*. For all* $\hat{\tau} > 0$ *there exists a number* $\hat{\epsilon} > 0$ *such that* h *has a unique zero in* Ω *for all* $(\tau, \epsilon) \in [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}]$*. Furthermore, there is a unique function* $r :\ [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}] \rightarrow \mathbb{C}$*,* $(\tau, \epsilon) \mapsto r(\tau, \epsilon)$*, which satisfies* $r(0, 0) = j\Omega$*,*

$$h(r(\tau, \epsilon);\ \tau, \epsilon) = 0, \ \ \forall (\tau, \epsilon) \in [0,\ \hat{\tau}] \times [-\hat{\epsilon},\ \hat{\epsilon}],$$

and which can be decomposed as:

$$r(\tau, \epsilon) = \hat{s} + \epsilon\ \mu(\tau, \epsilon),$$

where

$$\lim_{|\epsilon| \rightarrow 0+} \max_{\tau \in [0,\ \hat{\tau}]} \left| \mu(\tau, \epsilon) + \frac{aj\Omega + e^{-j\Omega\tau}}{2j\Omega} \right| = 0. \tag{42}$$

According to (42), for small values of ϵ the zero of $h(s;\ \tau, \epsilon)$ that correspond to the zero $j\Omega$ of $h(s;\ \tau, 0)$, behaves as the function

$$\tau \mapsto j\Omega - \frac{\epsilon(aj\Omega + e^{-j\Omega\tau})}{2j\Omega},$$

from which one can, for instance, conclude the following:

- if $a\Omega > 1$, then for all $\hat{\tau} > 0$, there is a number $\hat{\epsilon} > 0$ such that the zeros of $h(s;\ \tau, \epsilon)$ are in $\mathbb{C}_-$ for all $\tau \leq \hat{\tau}$ and $\epsilon \in (0,\ \hat{\epsilon})$;
- if $a\Omega < -1$, then for all $\hat{\tau} > 0$, there is a number $\hat{\epsilon} > 0$ such that $h(s;\ \tau, \epsilon)$ has zeros in $\mathbb{C}_+$ for all $\tau \leq \hat{\tau}$ and $\epsilon \in (0,\ \hat{\epsilon})$;
- if $a\Omega \in (-1,\ 1)$, then for all $\hat{\tau} > 0$, there is a number $\hat{\epsilon}$ and a number $\tau > \hat{\tau}$ such that for all $\epsilon \in (0,\ \hat{\epsilon})$ the zeros of $h(s;\ \tau, \epsilon)$ are in $\mathbb{C}_-$.

Following the lines of Section 3 the zeros of a more general function of the form (5), where f has more than one rightmost zero on the imaginary axis, can be analyzed. This can be done by first studying the behavior of the critical zeros separately, as illustrated above. Making assertions about the behavior of all critical zeros in the next step, such as stability assertions, then again leads to 'synchronization' requirements on the delay parameter.

6 Concluding remarks

From the study of the behavior of the zeros of (1) as a function of the delay parameter, insight was gained in the effect of time-delays on the stability of oscillatory systems and in the stabilization mechanism of such systems using delayed feedback. The results were supported by characteristic roots computations and validated by means of examples where analytic expressions for the stability regions can be obtained.

Since Proposition 3.1 is inherently a low gain type result (there is no lower bound on the gain parameter in the stability condition), we believe that by an appropriate gain scheduling technique the result can be extended towards the global stabilization of oscillatory systems with constraints on the input. A result in this direction is described in [18], where the global stabilization of a single oscillator with arbitrary input delay using state feedback is addressed.

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We like to thank the editors and the reviewers for their constructive comments and suggestion to analyze also the zeros of the function (41).

A Rouché's Theorem

The proofs of several results of the paper rely on the celebrated Rouché's Theorem (see, e.g. [16, §5.3.]):

Theorem A.1 *Let f and g be analytic functions on an (open) domain $\mathcal{D} \subseteq \mathbb{C}$. Let $\mathcal{C} \subset D$ be a closed, simple contour, that is, without self-interactions. If*

$$\forall s \in \mathcal{C}: \ |f(s) - g(s)| < |f(s)|,$$

then the functions f and, g have the same number of zeros inside $\mathcal{C}$, where each zeros is counted as many times as its multiplicity.

B Proofs

Proof of Proposition 4.3

The function (31) has a zero $j\omega,\ \omega > 0$, if and only if

$$\omega^4 - (\omega_1^2 + \omega_2^2)\omega^2 + \omega_1^2\omega_2^2 - \epsilon\omega^2 e^{-j\omega\tau} = 0.$$

Taking the real and imaginary parts of this equation yields

$$\begin{cases} \omega^4 - (\omega_1^2 + \omega_2^2)\omega^2 + \omega_1^2\omega_2^2 - \epsilon\omega^2\cos(\omega\tau) = 0, \\ \omega^2\sin(\omega\tau) = 0. \end{cases}$$

The solution of the second equation is given by either $\omega\tau = l\pi,\ l \in \mathbb{N}$, for which $\cos(\omega\tau) = 1$, or $\omega\tau = (2l+1)\pi,\ l \in \mathbb{N}$, for which $\cos(\omega\tau) = -1$. This allows to eliminate the delay τ in the first equation. In this way a straightforward computation yields (36) and (37). The condition (35) assures that the arguments of the square roots in (36) are positive.

If $\epsilon > 0$, respectively $\epsilon < 0$, then the condition (35) implies that

$$\tilde{\omega}_4 > \tilde{\omega}_3 > \tilde{\omega}_2 > \tilde{\omega}_1, \tag{43}$$

respectively,

$$\tilde{\omega}_3 > \tilde{\omega}_4 > \tilde{\omega}_1 > \tilde{\omega}_2. \tag{44}$$

From the crossing direction characterization of Theorem 6 of [22] and the results presented in [5], it follows that the crossing direction of a zero $j\omega,\ \omega > 0$, is towards instability if ω equals the largest frequency in (43)-(44), and alternates over the ordered frequencies. The assertion of the proposition follows.

Proof of Proposition 4.4

If $\epsilon > (\omega_2 - \omega_1)^2$, then $H(s;\ 0, \epsilon)$ has four zeros on the imaginary axis. Furthermore, the assertions of Proposition 4.3 remain valid provided (36) is replaced with the condition $\omega \in \{\tilde{\omega}_1, \tilde{\omega}_4\}$. Since zeros of H cross the imaginary axis towards stability for $\tau = 2\pi k/\hat{\omega}_1$ and towards instability for $\tau = 2\pi k/\hat{\omega}_4,\ k \in \mathbb{N}$, and since $\tilde{\omega}_1 < \tilde{\omega}_4$, stability cannot be obtained by increasing τ.

If $\epsilon < -(\omega_2 - \omega_1)^2$, then $H(s;\ 0, \epsilon)$ has 2 zeros in $\mathbb{C}_+$ and two zeros in $\mathbb{C}_-$. The assertions of Proposition 4.3 remain valid provided (36) is replaced with the condition $\omega \in \{\tilde{\omega}_2, \tilde{\omega}_3\}$. Since zeros of H cross the imaginary axis towards stability for $\tau = (2k+1)\pi/\hat{\omega}_2$ and towards instability if $\tau = (2k+1)\pi/\hat{\omega}_3,\ k \in \mathbb{Z}$, and since $\tilde{\omega}_2 < \tilde{\omega}_3$, again stability cannot be obtained by increasing τ.

The proof for the limit case $|\epsilon| = (\omega_2 - \omega_1)^2$ is by contradiction. Assume for instance that $H(s; \hat{\tau};\ (\omega_2 - \omega_1)^2)$ has all zeros in $\mathbb{C}_-$. Then for sufficiently small values of $\gamma > 0$, $H(s;\ \hat{\tau}, (\omega_2 - \omega_1)^2 + \gamma)$ must have all zeros in $\mathbb{C}_-$, and we have arrived at a contradiction.

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