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Stability analysis of systems with stochastically varying delays[☆]

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ABSTRACT

A stability analysis of linear systems with stochastically varying delay is performed. It is assumed that the delay function has the form of a sawtooth with switches occurring at the arrival times of a homogeneous Poisson process. Several notions of stochastic stability are considered and corresponding stability criteria are derived. For two examples the different criteria are compared and the effect on stability of various deterministic approximations is examined.

1. Introduction

In this note we derive stability criteria for linear systems subjected to a specific class of stochastically varying delays. Unlike the mainstream perturbation based approaches, wherein the analysis a system with time-varying delay is implicitly seen as a perturbation of a system with a time-invariant delay, we aim at exploiting explicitly the distribution and probabilistic properties of the delay. This allows us to determine the precise effect of the delay variation on stability, and, in particular, to characterize situations where the variation of the delay is beneficial and where it is not. From an application point of view the problem is motivated by networked controlled systems and congestion control problems in communication networks, where the delays typically have an irregular behavior, but information is available about their probabilistic properties [1,2].

While most of the existing stability criteria for systems with time-varying delays are based on viewing a time-varying delay as a perturbation of constant delay, there are a few results on assessing the precise effects of a delay variation on stability, yet for the special case of a (deterministic) periodic variation, see, e.g., [3–5]. In the latter reference it is shown that the stability properties of a system with a fast periodically varying delay are closely related to those of a comparison system with distributed delay, where the delay kernel can be interpreted as the distribution of the original time-varying delay. The investigation to what extent this result extends toward systems with stochastically varying delays is another motivation for the present paper. Here we note that the recent work on distributed delays with a so-called γ-distribution kernel in, e.g., [6,2] and the references therein is also inspired by problems with time-varying or stochastic delays. On the other hand, several results are available on stability of stochastic systems with delay, see [7] for an overview, but they do not cover the case where precisely the delay variation is stochastic. Exceptions are formed by the paper [8], where the available information about the delay is an upper bound which can take two values with a given probability, and by the book [9] where jump linear systems with mode dependent delays are briefly considered.

Because of the difficulty of the important problem of analyzing systems with general stochastically varying delays and the fact that this research direction is merely explored, we restrict ourselves in this work to the simplified case where the stochastic delay function has the shape of a sawtooth, with discontinuities occurring at the arrival times of a homogeneous Poisson process. The slope of the sawtooth is chosen equal to one, which not only simplifies the analysis, but allows us to interpret the problem and the obtained criteria in the context of stability of products of random matrices. Furthermore, the inspiring paper [5] suggests to consider a fast delay variation and thus a high slope, but a slope larger than one is debatable from a system's theoretic point of view, as a system with time-varying delay $\tau(t)$ may not be causal if the function $t-\tau(t)$

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is allowed to decrease [10]. Indeed, if the rate of the delay variation exceeds one, the system has to remember what it already forgot, which can then only be done by integrating the delay equation backwards.

As already mentioned, part of our results are based upon a fundamental theorem on products of random matrices [11,12], reproduced below to make the paper self-contained.

Theorem 1 (*Furstenberg–Kesten*). *If $\{A_k\}$ is a stationary ergodic sequence in $\mathbb{R}^{n\times n}$, for which $\mathbf{E}\max\{\log\|A_1\|\}<\infty$, then*

$$\lim_{N\to\infty}\frac{1}{N}\log\|A_NA_{N-1}\cdots A_1\| = \lim_{N\to\infty}\frac{1}{N}\mathbf{E}\log\|A_NA_{N-1}\cdots A_1\|$$

with probability one.

This theorem finds applications in the study of almost sure convergence of some adaptive algorithms [13], and in the stability analysis of discrete time systems with multiplicative noise [14]. Much use has been made of the *multiplicative ergodic theorem* by Oseledec [15], which can be seen as a generalization of Theorem 1. We note that the results for products of random matrices have been extended further by Luenberger to a class of monotone nonlinear mappings [16].

The structure of the paper is as follows. In Section 2 the model is presented and briefly discussed. The main results are given in Sections 3–5 and concern stability criteria corresponding to different stability notions (moment stability, Lyapunov stability, convergence of all sample paths). The obtained results are illustrated in Section 6. A modification of the model is briefly discussed in Section 7 and some concluding remarks in Section 8 end the paper.

The following notation will be used. For a square matrix A, $\rho(A)$, respectively $\alpha(A)$ denote its spectral radius, respectively its spectral abscissa, that is, $\alpha(A) := \max_{\lambda\in\mathbb{C}}\{\Re(\lambda):\ \det(\lambda I-A)=0\}$. The expected value of x is denoted by $\mathbf{E}x$. Throughout the paper $\|\cdot\|$ stands for an arbitrary vector norm or its induced matrix norm.

Finally, we refer to [17] for various stochastic stability definitions. See also some recent results in [18,19].

2. Reduction to a discrete delay model

We consider the equation

$$\dot{x}(t) = A\,x(t) + B\,x(t-\tau(t)),\quad x(0)\in\mathbb{R}^n, \tag{1}$$

where the delay τ has a sawtooth form, as shown in Fig. 1. The slope is equal to one a.e. and the discontinuities (resets to zero) occur at time-instants

$$0<t_1<t_2<\cdots$$

that correspond to the arrival times of a homogeneous Poisson process with rate λ. Letting $t_0=0$, the inter-arrival times $\delta_k := t_{k+1}-t_k,\ k\geq 0$, are thus independent and identically distributed, and the probability density function of the inter-arrival time δ_k is given by

$$\mathcal{P}_{\delta_k}(t) = \lambda \mathrm{e}^{-\lambda t}.$$

Thus, the average inter-arrival time is $1/\lambda$ and the average delay $1/2\lambda$. Let

$$\Omega := \mathbb{R}_+\times\mathbb{R}_+\times\cdots$$

be the sample space of the inter-arrival times $\{\delta_1,\delta_2,\ldots\}$. For $\omega\in\Omega$ we denote the corresponding sample path of the delay function by $\tau(t;\ \omega)$. Similarly, with the function

$$t\in\mathbb{R}_+\mapsto x(t;\ y,\omega)$$

we denote the sample path, corresponding to ω, of the forward solution of (1) with initial condition $x(0)=y$. If no confusion is possible, we will write $\tau(t)$ and $x(t)$ instead of $\tau(t;\ \omega)$ and $x(t;\ y,\omega)$ in what follows, in order to simplify the notation.

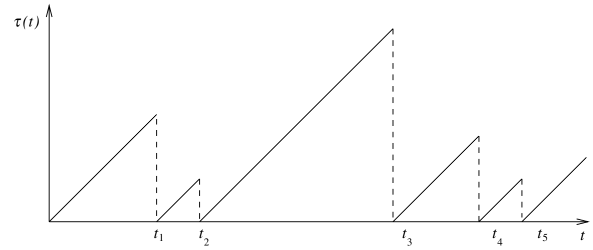

Fig. 1. Sample path for the stochastic delay function.

The delay function is such that

$$t-\tau(t)=t_k,\quad \forall t\in[t_k,\ t_{k+1}).$$

Therefore, if $t_k\leq t<t_{k+1}$ then Eq. (1) reduces to

$$\dot{x}(t) = A\,x(t) + B\,x(t_k), \tag{2}$$

which is readily integrated to

$$\begin{aligned} x(t) &= \mathrm{e}^{A(t-t_k)}x(t_k) + \int_{t_k}^{t}\mathrm{e}^{A(t-\theta)}Bx(t_k)\,\mathrm{d}\theta \\ &= \mathrm{e}^{A(t-t_k)}x(t_k) + \left(\mathrm{e}^{A(t-t_k)}-I\right)A^{-1}Bx(t_k). \end{aligned} \tag{3}$$

The integral is always defined, so $(\mathrm{e}^{A(t-t_k)})A^{-1}$ is to be understood as the convergent series expansion which exists, even if A is singular. At the next Poisson arrival time we get

$$x(t_{k+1}) = \mathrm{e}^{A(t_{k+1}-t_k)}x(t_k) + \left(\mathrm{e}^{A(t_{k+1}-t_k)}-I\right)A^{-1}Bx(t_k). \tag{4}$$

Denoting $x(t_k)$ simply by x_k, we arrive at the discrete model

$$x_{k+1} = M(\delta_k)\,x_k, \tag{5}$$

where the iteration matrix M satisfies

$$M(t) = \mathrm{e}^{At}(I+C)-C,\quad t\geq 0, \tag{6}$$

with

$$C = A^{-1}B.$$

Again we remark that $M(t)$ exists, although C may not.

Note that the existence and uniqueness of solutions is guaranteed by the relation between (1) and (5), combined with the existence of uniqueness of the solutions of (2) in the intervals $[t_k,\ t_{k+1})$.

3. Moment stability

Taking expectations of the iteration (5), we get due to the independence of δ_k that

$$\begin{aligned} \mathbf{E}\,x_{k+1} &= \mathbf{E}M(\delta_k)\,\mathbf{E}x_k \\ &= \int_0^\infty M(t)\lambda\mathrm{e}^{-\lambda t}\mathrm{d}t\,\mathbf{E}x_k \\ &= (\lambda I-A)^{-1}(\lambda I+B)\,\mathbf{E}x_k, \end{aligned} \tag{7}$$

under the condition $\lambda>\alpha(A)$, which assures the existence of the integral. Consequently, if the eigenvalues of the bilinear function $(\lambda I-A)^{-1}(\lambda I+B)$ lie within the unit circle, then the first moment converges to zero exponentially:

Theorem 2. *There exist constants $c>0$ and $\gamma\in(0,\ 1)$ such that*

$$\|\mathbf{E}x_k\|\leq c\gamma^k\,\|x_0\|$$

if and only if $\lambda>\alpha(A)$ and, in addition,

$$\rho\left((\lambda I-A)^{-1}(\lambda I+B)\right)<1. \tag{8}$$

Next we consider the stability of the second moment. From (5) we have

$$x_{k+1}x_{k+1}^{\mathrm{T}} = M(\delta_k)x_k x_k^{\mathrm{T}} M(\delta_k)^{\mathrm{T}},$$

hence,

$$\begin{aligned}\mathbf{E}x_{k+1}x_{k+1}^{\mathrm{T}} &= \mathbf{E}M(\delta_k)x_k x_k^{\mathrm{T}} M(\delta_k)^{\mathrm{T}} \\ &= \mathbf{E}\left\{M(\delta_k)(\mathbf{E}x_k x_k^{\mathrm{T}})M(\delta_k)^{\mathrm{T}}\right\}.\end{aligned}$$

If we define

$$P_k = \mathbf{E}x_k x_k^{\mathrm{T}}, \tag{9}$$

then we arrive at the iteration

$$P_{k+1} = \mathbf{E}\{M(\delta_k)P_k M(\delta_k)^{\mathrm{T}}\}. \tag{10}$$

Vectorizing this expression yields:

$$\operatorname{vec} P_{k+1} = \mathbf{E}\,(M(\delta_k)\otimes M(\delta_k))\operatorname{vec} P_k,$$

or

$$\operatorname{vec} P_{k+1} = U \operatorname{vec} P_k, \tag{11}$$

where

$$U := \mathbf{E}(M(\delta_k)\otimes M(\delta_k)) = \int_0^\infty M(t)\otimes M(t)\,\lambda \mathrm{e}^{-\lambda t}\mathrm{d}t. \tag{12}$$

Note that the integral exists if $\lambda > 2\alpha(A)$. From (11) we obtain:

Theorem 3. *There exist constants $c > 0$ and $\gamma \in (0, 1)$ such that*

$$\|\mathbf{E}x_k x_k^{\mathrm{T}}\| \le c\gamma^k \|x_0 x_0^{\mathrm{T}}\|$$

if and only if $\lambda > 2\alpha(A)$ and, in addition,

$$\rho(U) < 1, \tag{13}$$

where U is defined in (12).

Remark. Under the assumption that A is nonsingular, we obtain from (10):

$$\begin{aligned}P_{k+1} &= \mathbf{E}\,M(\delta_k)P_k M(\delta_k)^{\mathrm{T}} \\ &= \mathbf{E}\left(\mathrm{e}^{A\delta_k}(I+C)-C\right)P_k\left(\mathrm{e}^{A^{\mathrm{T}}\delta_k}(I+C^{\mathrm{T}})-C^{\mathrm{T}}\right).\end{aligned} \tag{14}$$

By defining

$$X_k := \frac{1}{\lambda}\,\mathbf{E}\,\mathrm{e}^{A^{\mathrm{T}}\delta}(I+C)P_k(I+C^{\mathrm{T}})\mathrm{e}^{A\delta},$$

which satisfies the Lyapunov equation

$$AX_k + X_k A^{\mathrm{T}} - \lambda X_k = (I+C)P_k(I+C^{\mathrm{T}}), \tag{15}$$

we can rewrite (14) as

$$\begin{aligned}P_{k+1} &= \lambda X_k - \lambda C P_k(I+C^{\mathrm{T}})(\lambda I - A^{\mathrm{T}})^{-1} \\ &\quad - \lambda(\lambda I - A)^{-1}(I+C)P_k C^{\mathrm{T}} + C P_k C^{\mathrm{T}}.\end{aligned} \tag{16}$$

The expressions (15) and (16) define a *two-term recursion* for the iteration (10) or (11). This iteration can be used in a power method for the computation of $\rho(U)$ in the application of Theorem 3. ◇

4. Lyapunov stability

Consider for the system (5)–(6) the Lyapunov function

$$V(x) = \mathbf{E}\,x^{\mathrm{T}}Px.$$

Note that for a given value of x_0, we get

$$\begin{aligned}V(x_1) &= \mathbf{E}\,x_1^{\mathrm{T}}Px_1 \\ &= \mathbf{E}\,x_0^{\mathrm{T}}M(\delta_0)^{\mathrm{T}}PM(\delta_0)x_0 \\ &= x_0^{\mathrm{T}}\left(\int_0^\infty M(t)^{\mathrm{T}}PM(t)\lambda \mathrm{e}^{-\lambda t}\mathrm{d}t\right)x_0.\end{aligned}$$

In order to assure the boundedness of the integral and a finite value of $V(x_1)$, the assumption $\lambda > 2\alpha(A)$ is made in the remainder of the section.

Letting $V_k = V(x_k)$, the increment $\Delta V_k := V_{k+1} - V_k$, satisfies

$$\begin{aligned}\Delta V_k &= \mathbf{E}\,x_k^{\mathrm{T}}[((I+C^{\mathrm{T}})\mathrm{e}^{A^{\mathrm{T}}\delta_k} - C^{\mathrm{T}})P(\mathrm{e}^{A\delta_k}(I+C)-C)-P]x_k \\ &= \mathbf{E}\,\Big\{x_k\Big[(I+C^{\mathrm{T}})[\mathbf{E}\,\mathrm{e}^{A^{\mathrm{T}}\delta_k}P\mathrm{e}^{A\delta_k}](I+C) \\ &\quad - \lambda C^{\mathrm{T}}P(\lambda I - A^{\mathrm{T}})^{-1}(I+C) \\ &\quad - \lambda(I+C^{\mathrm{T}})P(\lambda I - A)^{-1}C + C^{\mathrm{T}}PC - C\Big]x_k\Big\}.\end{aligned}$$

The matrix

$$X_P := \frac{1}{\lambda}\,\mathbf{E}\,\mathrm{e}^{A^{\mathrm{T}}\delta_k}P\mathrm{e}^{A\delta_k} = \int_0^\infty \mathrm{e}^{A^{\mathrm{T}}t}P\mathrm{e}^{At}\,\mathrm{e}^{-\lambda t}\mathrm{d}t,$$

satisfies the Lyapunov equation

$$A^{\mathrm{T}}X_P + X_P A - \lambda X_P + P = 0,$$

hence, also

$$(A^{\mathrm{T}} - \lambda I)X_P + X_p(A - \lambda I) + \lambda X_P + P = 0.$$

Substituting these expressions the increment becomes

$$\begin{aligned}\Delta V_k &= \mathbf{E}\,\Big\{x_k\Big[\lambda(I+C^{\mathrm{T}})X_P(I+C) - \lambda C^{\mathrm{T}}P(\lambda I - A)^{-1}(I+C) \\ &\quad - \lambda(I+C^{\mathrm{T}})(\lambda I - A)^{-1}PC + C^{\mathrm{T}}PC - C\Big]x_k\Big\}.\end{aligned} \tag{17}$$

Let Q be defined by

$$Q := (I+C^{\mathrm{T}})^{-1}C^{\mathrm{T}}P(\lambda I - A)^{-1} + (\lambda I - A^{\mathrm{T}})^{-1}PC(I+C)^{-1}.$$

As it satisfies

$$\begin{aligned}&(\lambda I - A^{\mathrm{T}})(I+C^{\mathrm{T}})^{-1}C^{\mathrm{T}}P + PC(I+C)^{-1}(\lambda I - A) \\ &\quad = (\lambda I - A^{\mathrm{T}})Q(\lambda I - A),\end{aligned}$$

we obtain

$$\begin{aligned}\Delta V_k &= \mathbf{E}\,x_k(I+C^{\mathrm{T}})\left[\lambda X_P - \lambda(I+C^{\mathrm{T}})C^{\mathrm{T}}P(\lambda I - A)^{-1}\right. \\ &\quad \left.- \lambda(\lambda I - A)^{-1}PC(I+C)^{-1}\right](I+C)x_k \\ &\quad + \mathbf{E}\,x_k^{\mathrm{T}}(C^{\mathrm{T}}PC - C)x_k \\ &= \mathbf{E}\,x_k\left[\lambda(I+C^{\mathrm{T}})(X_P - Q)(I+C) + C^{\mathrm{T}}PC - P\right]x_k.\end{aligned}$$

If the condition

$$\lambda(I+C^{\mathrm{T}})(X_P - Q)(I+C) + C^{\mathrm{T}}PC - P < 0 \tag{18}$$

is satisfied, then the system is stochastically Lyapunov stable, since (18) implies

$$V_{k+1} - V_k \le -\gamma V_k,$$

where

$$\gamma = \lambda_{\min}\left(-\lambda(I+C^{\mathrm{T}})(X_P - Q)(I+C) - C^{\mathrm{T}}PC + P\right) \in (0, 1).$$

Summing up the above results we arrive at the following theorem:

Theorem 4. *If $\lambda > 2\alpha(A)$ and if there exist matrices $P = P^T > 0$, $Q = Q^T$ and $X_P = X_P^T$ such that the following linear conditions are satisfied:*

$$\begin{aligned} &\lambda(I + C^T)(X_P - Q)(I + C) + C^T PC - P < 0, \\ &(\lambda I - A^T)(I + C^T)^{-1} C^T P + PC(I + C)^{-1}(\lambda I - A) \\ &\quad = (\lambda I - A^T) Q (\lambda I - A), \\ &A^T X_P + X_P A - \lambda X_P + P = 0, \end{aligned} \tag{19}$$

then the system (5) and (6) is stochastically Lyapunov stable. Furthermore, a Lyapunov function is given by $V = \mathbf{E}\, x^T P x$.

Remark. In the deterministic case, where the equation is governed by

$$x_{k+1} = M x_k, \tag{20}$$

with M independent of k, it is well known that Lyapunov stability and second moment stability are equivalent (for the latter note that $\rho(M \otimes M) < 1 \Leftrightarrow \rho(M) < 1$). Whether this property carries over to the stochastic case, more precisely, whether the feasibility of (19) is equivalent with the condition (13), is an open problem and subject of further research. ◇

5. Multiplicative ergodic theorem

Let $q \in \mathbb{N}_0$. Performing $k\,q$ iterations of (5) yields

$$x_{kq} = \prod_{l=1}^{k} N(\delta_{(l-1)q+1}, \ldots, \delta_{lq}) x_0,$$

where

$$N(s_1, \ldots, s_q) = M(s_q) M(s_{q-1}) \cdots M(s_1).$$

It follows that

$$\|x_{kq}\| \le \prod_{l=1}^{k} \|N(\delta_{(l-1)q+1}, \ldots, \delta_{lq})\| \, \|x_0\|$$

and

$$\frac{1}{k} \log \|x_{kq}\| \le \frac{1}{k} \sum_{l=1}^{k} \log \|N(\delta_{(l-1)q+1}, \ldots, \delta_{lq})\| + \frac{1}{k} \log \|x_0\|.$$

Every solution of (1) converges to zero if

$$\lim_{k\to\infty} \frac{1}{k} \log \|x_{kq}\|$$

exists and is strictly negative. An application of Theorem 1 yields:

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{k} \sum_{l=1}^{k} \log \|N(\delta_{(l-1)q+1}, \ldots, \delta_{lq})\| \\ &\quad = \mathbf{E} \log \|N(\delta_{(l-1)q+1}, \ldots, \delta_{lq})\| \\ &\quad = \int_0^\infty \cdots \int_0^\infty \log \|N(s_1, \ldots, s_q)\| \lambda^q e^{-\lambda(s_1 + \cdots + s_q)} ds_1 \ldots ds_q. \end{aligned}$$

Thus, we arrive at the following criterion:

Theorem 5. *Assume that $q \in \mathbb{N}_0$. If*

$$\int_0^\infty \cdots \int_0^\infty \log \|M(s_q) \cdots M(s_1)\| \lambda^q \times e^{-\lambda(s_1 + \cdots + s_q)} ds_1 \ldots ds_q < 0, \tag{21}$$

then with probability one, we have

$$\lim_{t\to\infty} x(t; y, \omega) = 0, \quad \forall y \in \mathbb{R}^n, \forall \omega \in \Omega.$$

Remark. If (21) is satisfied for $q = 1$, then it is also satisfied for all $q > 1$. Hence, taking $q > 1$ may relax the stability condition, at the price of an increased computational cost to check the criterion. ◇

Remark. When applied to the (deterministic) Eq. (20) the condition (21) becomes $\|M^q\| < 1$. This stability condition becomes necessary and sufficient in the limit case where $q \to \infty$, following from

$$\lim_{q\to\infty} \|M^q\| = (\rho(M))^q. \quad ◇$$

6. Example case studies

6.1. Scalar and triangularizable systems

We consider the equation

$$\dot{x}(t) = a x(t) + b x(t - \tau(t)), \quad a, b \in \mathbb{R}, \tag{22}$$

where τ is a sawtooth function, as described in Section 2. The switching is determined by a Poisson process with rate λ, or, equivalently, the average value of the delay, τ^{av}, satisfies

$$\tau^{av} = \frac{1}{2\lambda}. \tag{23}$$

In order to analyze the stability in the (a, b, τ^{av}) parameter space, we perform a transformation of time

$$t^{(new)} = t^{(old)} / \tau^{av}, \tag{24}$$

which brings the equation in the form

$$\dot{x}(t) = \alpha x(t) + \beta x(t - r(t)), \tag{25}$$

where

$$\alpha = a\tau^{av}, \quad \beta = b\tau^{av}$$

and

$$r(t) = \tau\left(\tau^{av} t\right) / \tau^{av}. \tag{26}$$

Note that the discontinuities of the normalized function r are determined by a Poisson process with rate 1/2. Hence only two effective parameters remain.

An application of Theorem 2 leads to the following stability condition for the first moment:

$$\alpha - 1 < \beta < -\alpha.$$

By Theorem 3 the second moment is stable if

$$\left| \int_0^\infty \left(e^{\alpha t} (1 + \beta/\alpha) - \beta/\alpha \right)^2 \frac{1}{2} e^{-\frac{1}{2}t} dt \right| < 1.$$

A simple calculation yields that this condition is equivalent with

$$\alpha - \frac{1}{2} < \beta < -\alpha,$$

and, as the equation is scalar, the same result is obtained from Theorem 4. Finally, Theorem 5 states that all solutions of (22) converge to zero if[1]

$$\int_0^\infty \log \left| e^{\alpha t} (1 + \beta/\alpha) - \beta/\alpha \right| \frac{1}{2} e^{-\frac{1}{2}t} \, dt < 0. \tag{27}$$

This condition corresponds to a region in the (α, β) plane of the form

$$\gamma(\alpha) < \beta < -\alpha.$$

The function γ can be determined from (27) using numerical integration and a rootfinder. The result of this computation is shown

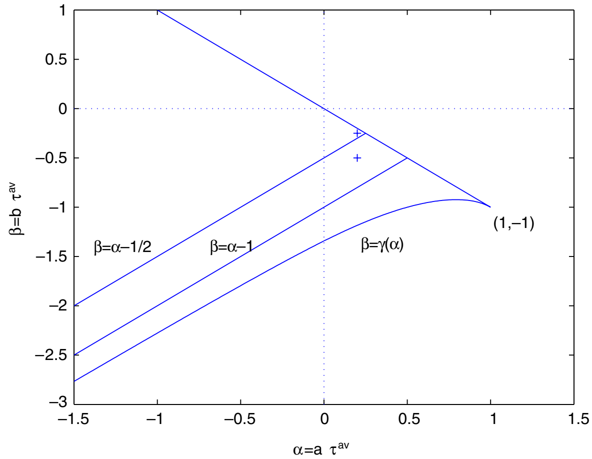

Fig. 2. Stability regions of (25) in the (a, b, τ^{av}) space, obtained from Theorem 2–3 and Theorem 5. For parameter values indicated with ('+') sample paths of solutions are shown in Fig. 3.

on Fig. 2, where we also compare the different stability criteria. It is important to point out that these criteria are non-conservative in the scalar case considered. The implication that for some parameter values every sample path converges to zero while the second moment is unstable, i.e. $\lim_{k\to\infty} \mathbf{E}x_k^2 = \infty$, is clarified by means of Fig. 3. In the top frame various sample paths of solutions of (22) are shown for parameter values $(\alpha, \beta) = (0.2, -0.25)$, for which both the second moment is stable and condition (21) is satisfied. Note that the convergence is uniform. The bottom frame corresponds to parameter values $(\alpha, \beta) = (0.2, -0.5)$, for which the second moment is unstable, but condition (21) is still satisfied. Although all sample paths converge to zero, transient behavior characterized by high peaks may occur. For larger values of k such peaks are less likely to occur, but the high values make $\mathbf{E}x_k^2$ grow unbounded.

Next, we have compared the obtained results with various results based on averaging the delay. More precisely the following comparison systems were considered: the equation

$$\dot{x}(t) = \alpha x(t) + \beta x(t - s(t)), \tag{28}$$

with

$$s(t) = t - 2n, \quad t \in [2n, 2(n+1)), n \in \mathbb{Z},$$

is obtained by replacing the Poisson inter-arrival times with their expected value. The equation

$$\dot{x}(t) = \alpha x(t) + \beta x(t-1) \tag{29}$$

is the result of averaging the whole delay function over time, following from

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T r(s;\omega)\mathrm{d}s = 1,$$

independent of ω. Finally, in

$$\dot{x}(t) = \alpha x(t) + \beta \int_0^\infty w(\theta)x(t-\theta)\mathrm{d}\theta, \tag{30}$$

where

$$w(\theta) = \frac{1}{2}\int_{\frac{\theta}{2}}^\infty \frac{\mathrm{e}^{-s}}{s}\,\mathrm{d}s,$$

[1] In the scalar case the condition (21) is independent of q.

Fig. 3. Plot of x_k versus k, corresponding to 50 sample paths of solutions of (25) with initial condition $x(0) = 1$. The parameter values are $(\alpha, \beta) = (0.2, -0.25)$ (top frame) and $(\alpha, \beta) = (0.2, -0.5)$ (bottom frame).

the time-varying delay $r(t)$ is replaced with a time-invariant, distributed delay whose kernel w is such that

$$\int_0^s w(\theta)\mathrm{d}\theta = \lim_{T\to+\infty}\frac{1}{T}\mu\left\{t \in [0,\ T] : r(t;\ \omega) < s\right\}, \quad \forall s \geq 0, \tag{31}$$

with $\mu(\cdot)$ the Lebesgue measure over $\mathbb{R}$. The latter approach is inspired by the paper [5], where for the case of a fast periodic delay variation it is shown that a distributed delay comparison system based on the relation (31) has strongly connected stability properties.

In Fig. 4 we have depicted the stability regions in the (α, β) space of (28)–(30) and compared with these of the original system (25), shown in Fig. 2. The unboundedness of the delay r and the relatively slow variation compared to the system's dynamics have contributed to the significant differences (for comparison, a small delay perturbation around a nominal value would have favored the approximation (29), while a very fast variation would have favored (30)). This illustrates the importance of taking into account the precise properties of the stochastic delay variation.

Remark. If A and B in (1) are both upper or lower triangular, then the structure of the equation suggests to split up the stability analysis in scalar problems

$$\dot{y}_i(t) = a_{i,i}\, y_i(t) + b_{i,i}\, y_i(t - \tau(t)), \quad i = 1, \ldots, n, \tag{32}$$