

THE SMOOTHED SPECTRAL ABCISSA FOR ROBUST STABILITY OPTIMIZATION*

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Abstract. This paper concerns the stability optimization of (parameterized) matrices $A(x)$, a problem typically arising in the design of fixed-order or fixed-structured feedback controllers. It is well known that the minimization of the spectral abscissa function $\alpha(A)$ gives rise to very difficult optimization problems, since $\alpha(A)$ is not everywhere differentiable, and even not everywhere Lipschitz. We therefore propose a new stability measure, namely the *smoothed spectral abscissa* $\tilde{\alpha}_\epsilon(A)$, which is based on the inversion of a relaxed H_2 -type cost function. The regularization parameter ϵ allows tuning the degree of smoothness. For ϵ approaching zero, the smoothed spectral abscissa converges towards the non-smooth spectral abscissa from above, so that $\tilde{\alpha}_\epsilon(A) \leq 0$ guarantees asymptotic stability. Evaluation of the smoothed spectral abscissa and its derivatives w.r.t. the matrix parameters x can be performed at the cost of solving a primal-dual Lyapunov equation pair, allowing for an efficient integration into a derivative based optimization framework. Two optimization problems are considered: on the one hand the minimization of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(A(x))$ as a function of the matrix parameters for a fixed value of ϵ , and on the other hand the maximization of ϵ such that the stability requirement, $\tilde{\alpha}_\epsilon(A(x)) \leq 0$, is still satisfied. The latter problem can be interpreted as an H_2 -norm minimization problem, and its solution additionally implies an upper bound on the corresponding H_∞ -norm, or a lower bound on the distance to instability. In both cases additional equality and inequality constraints on the variables can be naturally taken into account in the optimization problem.

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1. Introduction. Stability optimization of linear and nonlinear continuous-time dynamic systems is both a highly relevant and a difficult task. The optimization parameters often stem from a feedback controller, which can be used to optimize either a performance criterion or the asymptotic stability around a certain steady state. When also robustness against perturbations of the system must be taken into account, the resulting optimization problem becomes even more challenging.

Assuming an adequate parameterization of the desired feedback controller, the problem of finding a suitable steady state along with a stabilizing feedback controller can essentially be transformed into a nonlinear programming problem. By collecting all optimization variables in a vector x , we can summarize the described stability optimization problem as

$$\min_x \Phi_{\text{stab}}(A(x)), \quad \text{s.t.} \quad g(x) = 0, \quad h(x) \leq 0, \quad (1.1)$$

where $A(x)$ is the system matrix depending smoothly on x , and the function $\Phi_{\text{stab}}(\cdot)$

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expresses our desire to optimize stability, under the given constraints. In the field of linear output feedback control, the closed-loop system matrix $A(x)$ will typically be of the form $A + BKC$, with A the open-loop system matrix, B and C the input and output matrices, and K containing the controller parameters x to be optimized.

The most straightforward choice for the objective function Φ_{stab} is related to the eigenvalues of A , namely the spectral abscissa $\alpha(A)$. This value is defined as the real part of the rightmost eigenvalue of the spectrum $\Lambda(A) = \{z \in \mathbb{C} : \det(zI - A) = 0\}$, that is, $\alpha(A) = \sup\{\Re(z) : z \in \Lambda(A)\}$.

The spectral abscissa is in general a non-Lipschitz and non-convex function of A [13, 14] and therefore typically a very hard function to optimize. Nonetheless, recent developments have led to algorithms that are able to tackle such non-smooth objective functions [8, 12, 25, 26]. The extension to infinite-dimensional systems has been made in [29]. Still, the spectral abscissa is also known to perform quite poorly in terms of robustness against parameter uncertainties. A tiny perturbation or disturbance to a parameter of a system that was optimized in the spectral abscissa can possibly lead to instability.

For this reason, more robust approaches have been proposed. Amongst those, the most prominent are H_∞ -optimization [1, 2, 7, 23, 24] and, closely related, the minimization of the pseudospectral abscissa [10, 28]. As these robust optimization formulations are connected to maximizing the distance to instability of the system under consideration, they inherently take the effect of perturbations into account in the stability measure. However, their objective functions still suffer from non-smoothness and associated high computational costs in optimization. Throughout this paper, we will use the standard notation α_ϵ for the pseudo-spectral abscissa, not to be confused with our symbol for the smoothed spectral abscissa, namely $\tilde{\alpha}_\epsilon$. Another, albeit less well-known robustness measure is the robust spectral abscissa, denoted by α_δ as in [8], and is based on Lyapunov variables.

The paper is organized as follows. In §2, we define the smoothed spectral abscissa and we outline its most important properties. Section 3 discusses how to efficiently compute this newly defined stability measure along with its derivatives. In §4, we explain how the smoothed spectral abscissa can be used to formulate optimization problems dealing with robust stability and §5 draws a relation with the pseudo-spectral abscissa. Finally, we illustrate our stabilization method by treating two numerical examples in §6.

2. The smoothed spectral abscissa. In this section we introduce the notion of the smoothed spectral abscissa as a new stability measure that is not susceptible to non-smoothness like the spectral abscissa and the H_∞ -norm are. It can in addition be attributed with certain beneficial robustness properties. We will use several well-known principles from robust control for linear systems such as stability, the H_2 -norm, controllability and observability. See e.g. [30] for an introduction. At the basis of the smoothed spectral abscissa lies the following stability criterion.

LEMMA 2.1. *For any submultiplicative matrix norm $\|\cdot\|$ the matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz stable if and only if the integral $\int_0^\infty \|\exp(At)\|^2 dt$ is finite.*

Proof. We proof both directions. Suppose $\int_0^\infty \|\exp(At)\|^2 dt$ is finite, then $\|\exp(At)\| \rightarrow 0$ for $t \rightarrow \infty$. It is well-known that for any norm $\|\cdot\|$ this is equivalent to $\alpha(A) < 0$, see e.g. [21]. Reversely, suppose that $\alpha(A) < 0$ and let $\|\cdot\|$ be any submultiplicative norm, then there exists $0 < \gamma < \infty$ such that $\|\exp(At)\| \leq \gamma \exp(\alpha(A)t/2)$ for all $t \geq 0$, see e.g. [16, Ch. 1 §3]. From this we can derive that $\int_0^\infty \|\exp(At)\|^2 dt \leq -\gamma^2/\alpha(A) < \infty$. \square

Inspired by this observation, we let $f : \mathbb{R}^{n \times n} \times \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$ be the matrix function that uses the Frobenius norm $\|M\|_F^2 := \text{trace}(M^T M)$ and that takes as its arguments, next to the matrix A , also a real-valued relaxation parameter s

$$f(A, s) := \int_0^\infty \|V e^{(A-sI)t} U\|_F^2 dt. \quad (2.1)$$

Here, the matrices U and V are to be seen as respective input and output weighting matrices, with (A, U) controllable and (V, A) observable. It is easy to see that $f(A, s)$ is nothing else than the squared weighted and relaxed H_2 -norm of a system with transfer function $\mathbf{H}_s(z) = V(zI - (A - sI))^{-1} U$, i.e.,

$$f(A, s) = \|\mathbf{H}_s\|_{\mathcal{H}_2}^2. \quad (2.2)$$

We continue with the following properties for the function $f(A, s)$.

LEMMA 2.2. $\forall A \in \mathbb{R}^{n \times n} : \{f(A, s) : s > \alpha(A)\} = \mathbb{R}^+ \setminus \{0\}$.

Proof. If $s > \alpha(A)$, the matrix $A - sI$ is stable and therefore $f(A, s)$ is finite by Lemma 2.1. Additionally, $f(A, s)$ tends to infinity and to zero for $s \rightarrow \alpha(A)$ and $s \rightarrow \infty$ respectively. \square

LEMMA 2.3. $\forall s > \alpha(A) : \partial f(A, s) / \partial s < 0$ and $\partial^2 f(A, s) / \partial s^2 > 0$.

Proof. This can be verified by differentiating the integral in equation (2.1) with respect to s once and twice respectively. \square

These last two properties allow us to introduce the implicit function of the relation $f(A, s) = \epsilon^{-1}$ w.r.t. the relaxation argument s , as it is well defined on the whole domain, that is, for any $\epsilon > 0$ and for any matrix $A \in \mathbb{R}^{n \times n}$. We will call this function the ‘‘smoothed spectral abscissa’’, analogously to the smoothed spectral radius for discrete time systems [15].

DEFINITION 2.4. *The smoothed spectral abscissa is defined as the mapping $\alpha : \mathbb{R}^{n \times n} \times \mathbb{R}^+ \setminus \{0\} \rightarrow \mathbb{R}$, $(A, \epsilon) \mapsto \tilde{\alpha}_\epsilon(A)$ that uniquely solves*

$$f(A, \tilde{\alpha}_\epsilon(A)) = \epsilon^{-1}. \quad (2.3)$$

Because $f(A, s)$ is analytic in both its arguments for any $s > \alpha(A)$, it follows from the implicit function theorem that $\tilde{\alpha}_\epsilon(A)$ is analytic on its whole domain $\epsilon > 0$, $A \in \mathbb{R}^{n \times n}$. Moreover, it has the following additional properties.

THEOREM 2.5. $\tilde{\alpha}_\epsilon(A)$ is an increasing function of ϵ , that is, $\partial \tilde{\alpha}_\epsilon(A) / \partial \epsilon > 0$.

Proof. Differentiating (2.3) on both sides w.r.t. ϵ , we obtain

$$\frac{df(A, \tilde{\alpha}_\epsilon(A))}{d\epsilon} = \frac{\partial f(A, s)}{\partial s} \frac{\partial \tilde{\alpha}_\epsilon(A)}{\partial \epsilon} = -\epsilon^{-2} < 0,$$

from which the proposition holds by Lemma 2.3. \square

THEOREM 2.6. $\forall \epsilon > 0 : \tilde{\alpha}_\epsilon(A) > \alpha(A)$ and $\lim_{\epsilon \rightarrow 0} \tilde{\alpha}_\epsilon(A) = \alpha(A)$.

Proof. These two properties follow from the fact that $f(A, s)$ is finite and descending for $s > \alpha(A)$, but tends to infinity as s approaches $\alpha(A)$. \square

Also note that this last theorem implies that a non-positive smoothed spectral abscissa guarantees that the underlying system is asymptotically stable. The above definition and properties are illustrated in Figure 2.1.

3. Computing the smoothed spectral abscissa and its derivatives. Having defined the smoothed spectral abscissa, we now take a look at its computation. As explained in the previous section, this involves solving the smooth but nonlinear

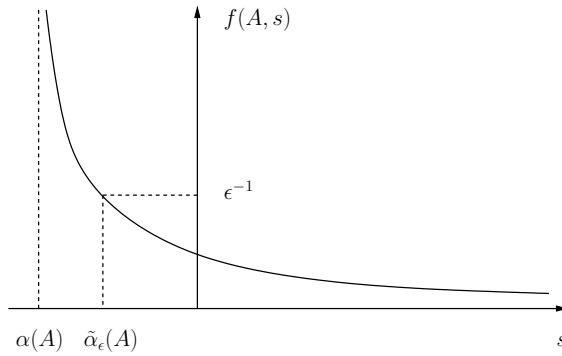


FIG. 2.1. Typical behaviour of the function $f(A, s)$ as a function of s . The smoothed spectral abscissa $\tilde{\alpha}_\epsilon(A)$ is the abscissa of the point where this function reaches ϵ^{-1} .

equation $f(A, s) = \epsilon^{-1}$ for s . Therefore, we first give some properties of the function $f(A, s)$ regarding its evaluation and its derivatives.

LEMMA 3.1. *For all $s > \alpha(A)$, there exist symmetric $n \times n$ matrices P and Q such that*

$$f(A, s) = \text{trace}(VPV^T) = \text{trace}(U^TQU), \quad (3.1a)$$

$$\frac{\partial f(A, s)}{\partial s} = -2 \text{trace}(QP) = -2 \text{trace}(PQ), \quad (3.1b)$$

$$\frac{\partial f(A, s)}{\partial A} = 2QP, \quad (3.1c)$$

where P and Q satisfy the primal-dual Lyapunov equation pair

$$0 = L(P, A, U, s), \quad (3.2a)$$

$$0 = L(Q, A^T, V^T, s), \quad (3.2b)$$

with L defined as

$$L(P, A, U, s) := (A - sI)P + P(A - sI)^T + UU^T.$$

Proof. The first part follows immediately by writing out the Frobenius norm in (2.1)

$$f(A, s) = \text{trace} \left(V \int_0^\infty e^{(A-sI)t} UU^T e^{(A-sI)^T t} dt V^T \right),$$

and by the well-known fact that, since $A - sI$ is stable, the above integral can be identified as the trace of P , the solution of equation (3.2a) (see, for instance, [18, 30]). Note that solving the dual Lyapunov equation (3.2b) computes a matrix Q that solves the dual integral

$$Q = \int_0^\infty e^{(A-sI)^T t} V^T V e^{(A-sI)t} dt.$$

Since A is fixed in the partial derivative $\frac{\partial f(A, s)}{\partial s}$, we can regard f as a function of P , where P depends on s through the Lyapunov relation $L(P(s), A, U, s)$. Rather than computing this partial derivative directly as

$$\frac{\partial f(A, s)}{\partial s} = \frac{d}{ds} \text{trace}(VP(s)V^T) = \text{trace} \left(V \frac{dP}{ds} V^T \right),$$

with $\frac{dP}{ds}$ the solution of the Lyapunov equation $(A - sI)\frac{dP}{ds} + \frac{dP}{ds}(A - sI)^T - 2P = 0$, we choose to use an adjoint differentiation technique. Vectorizing the matrix P in a $n^2 \times 1$ vector $p = \text{vec}(P)$, we can write

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial p} \frac{\partial p}{\partial s} = -\frac{\partial f}{\partial p} \left(\frac{\partial \ell}{\partial p} \right)^{-1} \frac{\partial \ell}{\partial s}, \quad (3.3)$$

where $\ell := \text{vec}(L(P, A, U, s))$ represents the vectorized primal Lyapunov equation (3.2a). Making use of the fact that $\text{vec}(MXN^T) = (N \otimes M) \text{vec}(X)$, where \otimes denotes the Kronecker product [17], we can make ℓ explicit in p , and as a result arrive at the following $n^2 \times n^2$ linear system

$$\ell(p, A, U, s) = \frac{\partial \ell}{\partial p} p + \text{vec}(UU^T) = 0,$$

with $\frac{\partial \ell}{\partial p} = (A - sI) \otimes I + I \otimes (A - sI)$. For the dual Lyapunov equation, we similarly obtain

$$\ell(q, A^T, V^T, s) = \frac{\partial \ell}{\partial q} q + \text{vec}(V^T V) = 0,$$

with $\frac{\partial \ell}{\partial q} = (A - sI)^T \otimes I + I \otimes (A - sI)^T$. It is easily verified that $\frac{\partial \ell}{\partial q} = \frac{\partial \ell}{\partial p}^T$. Replacing $\frac{\partial \ell}{\partial q}$ in the relation $\ell(q, A^T, V^T, s) = 0$, and using in addition the fact that $\text{vec}(V^T V)$ equals $\frac{\partial f}{\partial p}^T$, we find that

$$\frac{\partial \ell}{\partial p}^T q + \frac{\partial f}{\partial p}^T = 0 \quad \Leftrightarrow \quad q^T = -\frac{\partial f}{\partial p} \left(\frac{\partial \ell}{\partial p} \right)^{-1}.$$

Combining this with (3.3), along with $\frac{\partial \ell}{\partial s} = -2p$, finally gives

$$\frac{\partial f}{\partial s} = q^T(-2p) = -2 \text{vec}(Q)^T \text{vec}(P) = -2 \text{trace}(QP).$$

For the third part of the proof, i.e. the proof of the expression for the derivative w.r.t. A , we can use the same adjoint differentiation technique. Here, we again let f depend on the vectorized matrix $p = \text{vec}(P)$, which now depends on $a = \text{vec}(A)$ according to the relation $\ell(p(a), a, s) = 0$. Using the previous results, we obtain the following expression for $\frac{\partial f}{\partial a} := \text{vec}^T \left(\frac{\partial f}{\partial A} \right)$,

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial p} \frac{\partial p}{\partial a} = -\frac{\partial f}{\partial p} \left(\frac{\partial \ell}{\partial p} \right)^{-1} \frac{\partial \ell}{\partial a} = q^T \frac{\partial \ell}{\partial a}. \quad (3.4)$$

To find $\frac{\partial \ell}{\partial a}$, we first have to make ℓ explicit in a , which yields

$$\ell(P, a, U, s) = \frac{\partial \ell}{\partial a} a + \text{vec}(UU^T) = 0 \quad \text{with} \quad \frac{\partial \ell}{\partial a} = (P \otimes I) + (I \otimes P) \Pi, \quad (3.5)$$

where Π denotes the symmetric permutation matrix that satisfies $\text{vec}(A^T) = \Pi \text{vec}(A)$, e.g. $\Pi = S_{n,n}$ in [20]. Substituting in (3.4) gives

$$\text{vec}^T \left(\frac{\partial f}{\partial A} \right) = \left(\frac{\partial \ell}{\partial a}^T q \right)^T = [\text{vec}(QP) + \Pi^T \text{vec}(PQ)]^T = 2 \text{vec}^T(QP) = \text{vec}^T(2QP).$$

By comparison of both sides, we finally obtain that

$$\frac{\partial f}{\partial A} = 2QP,$$

which concludes the proof. \square

The relatively cheap computation of $f(A, s)$ and its derivative w.r.t. s enables us to efficiently solve the nonlinear equation $f(A, s) = \epsilon^{-1}$ by use of standard root finding methods, and thus evaluate the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(A)$. Specifically, we can use a Dekker–Brent type method [6] provided that we establish a root bracketing interval first, or Newton’s method if we want to exploit the availability of the derivatives. For further elaboration on the computational issues involving the smoothed spectral abscissa, see §6.3.

As we will want to use derivative-based optimization methods later on to exploit the smoothness of the smoothed spectral abscissa, we need to be able to compute also the derivative of $\tilde{\alpha}_\epsilon(A)$ w.r.t. A . Fortunately, this can be done at almost no extra cost. Indeed, the same ingredients that were needed for the evaluation of $\tilde{\alpha}_\epsilon(A)$, namely the solutions P and Q of one primal-dual Lyapunov equation pair, give us direct access to the derivative of $\tilde{\alpha}_\epsilon(A)$ w.r.t. A , as expressed in the following theorem.

THEOREM 3.2. *For fixed ϵ , the derivative of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(A)$ w.r.t. A equals*

$$\frac{\partial \tilde{\alpha}_\epsilon(A)}{\partial A} = \frac{QP}{\text{trace}(QP)},$$

where P and Q satisfy the Lyapunov equation pair (3.2a) – (3.2b) for $s = \tilde{\alpha}_\epsilon(A)$.

Proof. Differentiating the implicit equation $f(A, s) = \epsilon^{-1}$ w.r.t. A , and using the chain rule, we obtain

$$\frac{\partial \tilde{\alpha}_\epsilon(A)}{\partial A} = - \left(\frac{\partial f(A, s)}{\partial s} \right)^{-1} \left(\frac{\partial f(A, s)}{\partial A} \right).$$

Recalling (3.1b) and (3.1c) of Lemma 3.1, the result follows directly. \square

REMARK 1. Suppose A depends on an $m \times 1$ parameter vector x , then a direct approach to compute the derivatives w.r.t. to these parameters would require solving $m + 1$ Lyapunov equations with different right-hand sides, instead of $m + 1$ matrix multiplications of $\partial A / \partial x$ with $\partial \tilde{\alpha}_\epsilon / \partial A$.

4. Robust stability optimization. When it comes to algorithmic optimization, a first major advantage of the smoothed spectral abscissa criterion is that it is differentiable everywhere, and that its derivatives can be computed efficiently. This allows us to use derivative based methods without any restriction. Additionally, due to its differentiable dependence on A and its connection with the H_2 -norm, it is expected to be a more robust measure for stability than the spectral abscissa. We will present two smooth formulations of the stability optimization problem (1.1), one that focuses on mere stabilization, and one that will turn out to perform a H_2 -norm minimization.

The first variant is to simply choose a fixed $\epsilon > 0$ and then solve

$$\min_x \tilde{\alpha}_\epsilon(A(x)), \quad \text{s.t.} \quad g(x) = 0, \quad h(x) \leq 0. \quad (4.1)$$

Here, $\tilde{\alpha}_\epsilon(A(x))$ is indirectly dependent on the matrix parameter vector x , as it is implicitly defined as the solution of the relation $f(A(x), s) = \epsilon^{-1}$ w.r.t. s . By decoupling

this implicit relation into a constraint, we can formulate the problem alternatively as

$$\min_x s, \quad \text{s.t.} \quad f(A(x), s) = \epsilon^{-1}, \quad g(x) = 0, \quad h(x) \leq 0, \quad (4.2)$$

which is more amenable for an SQP optimization framework.

Should problem (4.1) or (4.2) not result in a negative optimal value for the chosen ϵ , then one can try again with a smaller ϵ . Note also that, if the sole goal is to achieve a stable system, one may terminate the optimization procedure once the smoothed spectral abscissa becomes smaller than zero.

In the minimization formulation of the smoothed spectral abscissa with fixed ϵ , the choice of ϵ is somewhat arbitrary. As indicated by Theorem 2.6, $\tilde{\alpha}_\epsilon(A)$ becomes smoother—and thus presumably a more robust measure for stability—with increasing values for $\epsilon > 0$. Thus, we might alternatively search for the largest ϵ so that the stability certificate $\tilde{\alpha}_\epsilon(A) \leq 0$ still holds. This leads to a second optimization problem

$$\max_{x, \epsilon} \epsilon \quad \text{s.t.} \quad \tilde{\alpha}_\epsilon(A(x)) \leq 0 \quad \text{and} \quad g(x) = 0, \quad h(x) \leq 0. \quad (4.3)$$

Since $\tilde{\alpha}_\epsilon(A)$ is a continuously growing function of ϵ , the constraint in problem (4.3) will always be active at its optimizer (x^*, ϵ^*) . Hence, it is easily seen that the solution of the first problem (4.1), with ϵ fixed to ϵ^* , will be exactly zero, and that in addition its minimizer will be the same as the one for problem (4.3), namely x^* . Succinctly,

$$x^* = \arg \min_x \tilde{\alpha}_{\epsilon^*}(A(x)) \quad \text{and} \quad \tilde{\alpha}_{\epsilon^*}(A(x^*)) = 0.$$

Problem (4.3) can thus be solved by finding the ϵ for which the resulting minimal smoothed spectral abscissa is zero, which can be implemented by bisecting with respect to ϵ . The activity of the stability constraint also leads to the following nice interpretation of problem (4.3).

THEOREM 4.1. *Any solution x^* that solves problem (4.3) also solves the H_2 -norm optimization of a system with transfer function $\mathbf{H}(x)(z) := V(zI - A(x))^{-1}U$, i.e.*

$$x^* = \arg \min_x \|\mathbf{H}(x)\|_{\mathcal{H}_2}, \quad \text{s.t.} \quad g(x) = 0, \quad h(x) \leq 0,$$

and the solution $\|\mathbf{H}(x^*)\|_{\mathcal{H}_2}$ is equal to $\sqrt{1/\epsilon^*}$.

Proof. Taking the inverse of the objective function in problem (4.3), and incorporating the fact that the stability constraint will be active, this problem can be rewritten as the minimization of the function $f(A(x), 0)$, subject to the constraints g and h , and additionally restricting x to values for which $A(x)$ is stable. This is, by (2.2), equivalent to minimizing the squared H_2 -norm of the system with transfer function \mathbf{H} . \square

REMARK 2. Solving problem (4.3) with the restriction $\tilde{\alpha}_\epsilon < s$ (with $s < 0$) would minimize the H_2 -norm of a system with the shifted transfer function \mathbf{H}_s .

5. Relation with the pseudo-spectral abscissa. We will now draw a relationship between the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(A)$ and the pseudo-spectral abscissa $\alpha_\epsilon(A)$, the latter being defined as

$$\alpha_\epsilon(A) := \sup\{\Re(z) : z \in \Lambda_\epsilon(A)\} \quad \text{where} \quad \Lambda_\epsilon(A) = \{\Lambda(X) : \|X - A\|_2 \leq \epsilon\}.$$

For this section we take the restriction $U = V = I$, so the transfer function becomes $\mathbf{H}(z) = (zI - A)^{-1}$. Define the H_∞ -norm as

$$\|\mathbf{H}\|_{\mathcal{H}_\infty} = \sup_{\Re(z)=0} \|\mathbf{H}(z)\|_2.$$

We then have the following well-known equivalency involving $\alpha_\epsilon(A)$ and the corresponding H_∞ -norm [10]

$$\alpha_\epsilon(A) < 0 \quad \Leftrightarrow \quad \|\mathbf{H}\|_{\mathcal{H}_\infty} < \epsilon^{-1}. \quad (5.1)$$

We can also interpret this in terms of the H_∞ -norm of a shifted matrix $A - sI$. The relation then becomes

$$\alpha_\epsilon(A - sI) < 0 \quad \Leftrightarrow \quad \alpha_\epsilon(A) < s \quad \Leftrightarrow \quad \|\mathbf{H}_s\|_{\mathcal{H}_\infty} < \epsilon^{-1}, \quad (5.2)$$

where $\mathbf{H}_s(z) := (zI - (A - sI))^{-1} = ((z + s)I - A)^{-1}$. In other words, the pseudo-spectral abscissa is the minimal shift-to-the-left s for which the ‘‘shifted’’ H_∞ -norm is smaller than ϵ^{-1} . Similarly as in Remark 2, it follows from (5.2) that the minimization of α_ϵ amounts to minimizing the H_∞ -norm of the shifted system \mathbf{H}_s where $s = \min \alpha_\epsilon$.

Going back to the definition of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon(A)$, and taking into account that f is a decreasing function of s (Lemma 2.3), we derive a similar relation as we did in (5.2)

$$\tilde{\alpha}_\epsilon(A) < s \quad \Leftrightarrow \quad f(A, s) = \|\mathbf{H}_s\|_{\mathcal{H}_2}^2 < \epsilon^{-1}. \quad (5.3)$$

Analogously, we can regard the smoothed spectral abscissa as the minimal shift s for which $\|\mathbf{H}_s\|_{\mathcal{H}_2}^2$ lies below the bound ϵ^{-1} (see also Figure 2.1). Thus, $\alpha_\epsilon(A)$ and $\tilde{\alpha}_\epsilon(A)$ are both relaxations of the spectral abscissa in the sense that they are both induced by placing a bound on a norm (H_∞ and H_2 respectively) that goes to infinity when approaching instability. This analogy enables us to relate these two robust stability measures.

THEOREM 5.1 (Relation to pseudo-spectral abscissa). *For $s > \alpha(A)$ and for $U = V = I$, the following holds*

$$\|\mathbf{H}_s\|_{\mathcal{H}_\infty} < 2\|\mathbf{H}_s\|_{\mathcal{H}_2}^2 \quad (5.4a)$$

$$\alpha_{\epsilon/2}(A) < \tilde{\alpha}_\epsilon(A). \quad (5.4b)$$

Proof. The first inequality is based on [3], where $2\lambda_{\max}(Q^2)^{\frac{1}{2}}$ is established to be an upper bound on the H_∞ -norm of an unweighted system with transfer function \mathbf{H}_s , where Q satisfies (3.2b). Since Q is a positive definite matrix, we can deduce from this the following,

$$\|\mathbf{H}_s\|_{\mathcal{H}_\infty} \leq 2\lambda_{\max}(Q) < 2\text{trace}(Q).$$

This proves (5.4a) directly by Lemma 3.1(a) and by (2.2). Suppose then, by (5.3), that for $s = \tilde{\alpha}_\epsilon(A)$ it is true that $\|\mathbf{H}_s\|_{\mathcal{H}_2}^2 = \epsilon^{-1}$. Using (5.4a) in connection with (5.2), assertion (5.4b) follows. \square

This property has an important implication in terms of robust optimization. It shows that the squared H_2 -norm constitutes an upper bound on the H_∞ -norm, which is directly related to the distance to instability of a system. By minimizing the first norm, one could expect that the second norm should also go down.

On top of this rather intuitive result, (5.4b) together with (5.1) provides us with a guarantee w.r.t. H_∞ -norm once the smoothed spectral abscissa is negative. Indeed, if we have that $\tilde{\alpha}_\epsilon(A(x)) < 0$ for some x , we are not only sure that the system with system matrix $A(x)$ will be a stable one, but also that this system will have an H_∞ -norm that is smaller than $2/\epsilon$. In other words, we can be certain that the distance to instability of the system will be at least $\epsilon/2$.

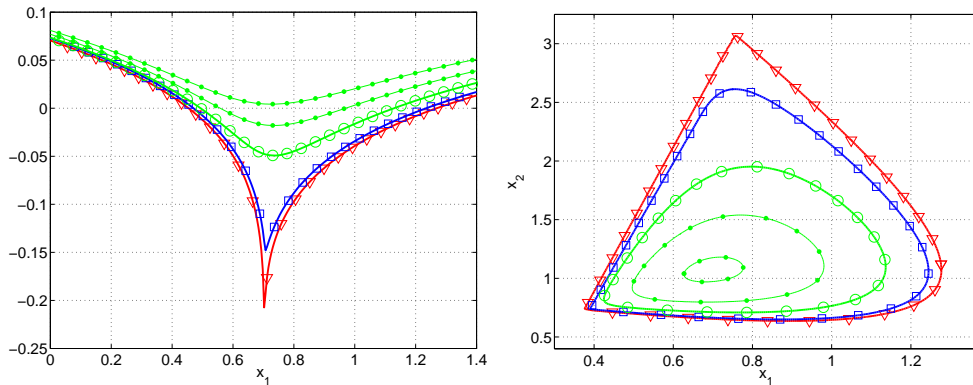


FIG. 6.1. Evolution with respect to x_1 (left) and stability regions (right) of the spectral abscissa α (with \blacktriangledown), pseudo-spectral abscissa $\alpha_{\epsilon/2}$ (with \square) and smoothed spectral abscissa $\tilde{\alpha}_\epsilon$ (with \circ) of the example in §6.1 with smoothing parameter $\epsilon = 4 \cdot 10^{-3}$. In addition (with \bullet), two smoothed spectral abscissae for $\epsilon = 8 \cdot 10^{-3}$ and $\epsilon = 12 \cdot 10^{-3}$.

6. Numerical examples. We will now put theory into practice by treating two control examples using the smoothed spectral abscissa as the stability criterion. First, we will illustrate the theory behind the smoothed spectral abscissa by use of an academic example. Next, we will treat a more realistic example, namely a turbo generator model. We conclude by making a comparison of the computational cost of the smoothed spectral abscissa in relation with other robust stability measures. All computations were done with MATLAB R2008a.

6.1. A simple state feedback controlled system. Consider the following two-parameter linear state feedback controlled system, with a closed-loop system matrix $A + BK$, and where

$$A = \begin{bmatrix} 0.1 & -0.03 & 0.2 \\ 0.2 & 0.05 & 0.01 \\ -0.06 & 0.2 & 0.07 \end{bmatrix}, \quad B = \frac{1}{2} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad K^T = \begin{bmatrix} x_1 \\ x_2 \\ 1.4 \end{bmatrix}. \quad (6.1)$$

Figure 6.1 shows, as a function of the control parameters, the spectral abscissa ($\epsilon = 0$) in comparison with the smoothed spectral abscissae for three different smoothing levels ($\epsilon = 4, 8, 12 \cdot 10^{-3}$). For $\epsilon = 4 \cdot 10^{-3}$, the corresponding pseudo-spectral abscissa, i.e. with an ϵ half as large, is also plotted. In the left frame $x_2 = 1.25$ is held fixed. In the right frame, both x_1 and x_2 are free and the boundaries of the stability regions, that is the regions where the respective measures are negative, are drawn. On both figures, we clearly observe the smooth behaviour of $\tilde{\alpha}_\epsilon$ in contrast with the non-smoothness of the spectral and pseudo-spectral abscissa.

The ordering of $\tilde{\alpha}_\epsilon$, $\alpha_{\epsilon/2}$ and α as stated by Theorems 2.6 and 5.1 is also confirmed. On the left, the curve of the smoothed spectral abscissa is everywhere above the other two curves, and on the right the $\tilde{\alpha}_\epsilon$ -stability region is strictly contained within the stability regions of the pseudo-spectral and consequently also of the spectral abscissa.

6.2. Turbo generator model. Next, we treat Problem ‘TG1’ of Leibfritz’s control problem database [19], which models a nuclear powered turbo generator by a linear system of dimension 10 with four control parameters. This system has already been used as an example in [9] for robust stability optimization using the pseudo-spectral abscissa in combination with the gradient sampling algorithm.

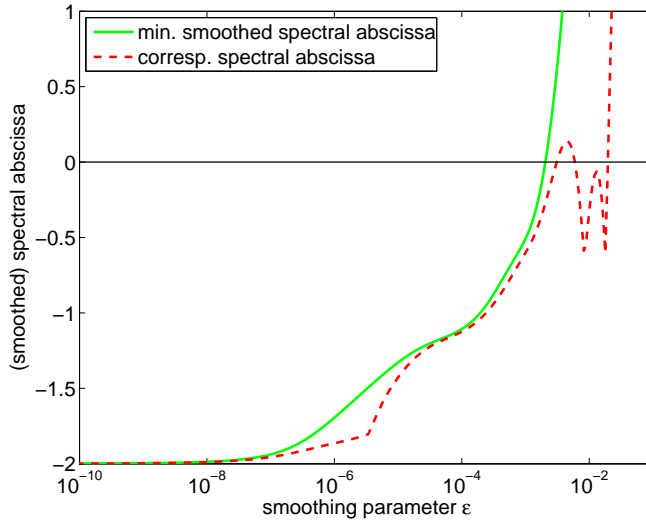


FIG. 6.2. The minimal smoothed spectral abscissa $\tilde{\alpha}_\epsilon$ of the turbo generator model, for ϵ ranging from 10^{-10} to 10^{-1} , and the corresponding spectral abscissa α evaluated in each of the minimizers.

$\log_{10} \epsilon$	$\min_x \tilde{\alpha}_\epsilon(x)$	iters	$\alpha_{\epsilon/2}(x^*)$	$\alpha(x^*)$
0	86.920558 / 86.920558	35 / 32	4.170208	9.899472
-1	23.951317 / 23.951317	37 / 33	5.267947	5.447648
-2	3.925292 / 3.925292	29 / 29	-0.327800	-0.272926
-3	-0.508181 / -0.508181	34 / 62	-0.600857	-0.598826
-4	-1.107119 / -1.107119	69 / 54	-1.125056	-1.124731
-5	-1.328287 / -1.328287	104 / 80	-1.427324	-1.426787
-6	-1.694445 / -1.694445	102 / 65	-1.864124	-1.864049
-7	-1.938475 / -1.938475	252 / 57	-1.955631	-1.955624
-8	-1.987303 / -1.987303	292 / 125	-1.988801	-1.988801
-9	-1.996336 / -1.996246	1522 / 51	-1.996542	-1.996542
-10	-1.998646 / -1.998587	1827 / 47	-1.998680	-1.998680

TABLE 6.1

Solutions to the minimization of the smoothed spectral abscissa of the turbo generator model for 11 designated ϵ -values (without homotopy/with homotopy), and the corresponding pseudo-spectral and spectral abscissae.

Figure 6.2 shows the behaviour of the solutions to a minimization of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon$ for a dense set of smoothing parameters ϵ between 10^{-1} and 10^{-10} , i.e. ranging from relatively large to very small. It is immediately verified that the minima of the smoothed spectral abscissa decrease monotonically for ϵ becoming smaller. Next to the minima, we also plotted the evolution of the corresponding spectral abscissae evaluated at each of these minimizers. We can see that, although α is always strictly smaller than $\tilde{\alpha}_\epsilon$, it is not guaranteed to decrease monotonically, which is for example the case for large ϵ . For $\epsilon \rightarrow 0$ however, the gap between the two becomes tighter and tighter. Of course, as the smoothed spectral abscissa converges to the spectral abscissa when ϵ approaches zero, the $\tilde{\alpha}_\epsilon$ -minimization problem becomes more non-smooth and thus harder.

To analyze this, Table 6.1 shows the results of a standard BFGS minimization

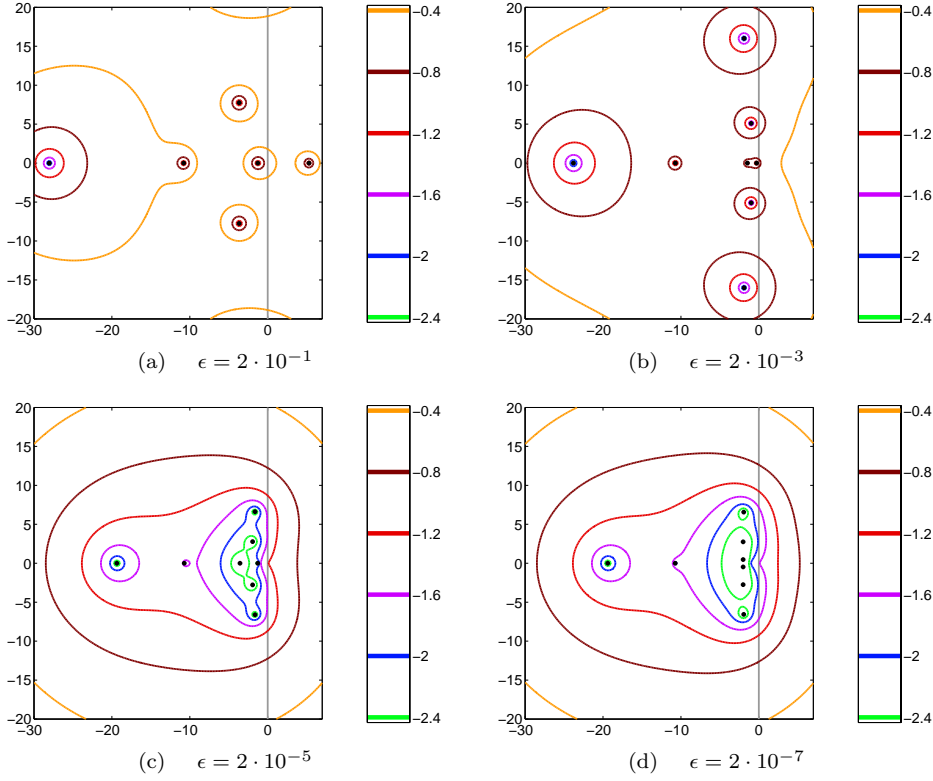


FIG. 6.3. Boundaries of the pseudospectra Λ_γ of the turbo generator model for values of γ as indicated in the right-hand side colour bar. The frames correspond to four sets of controller parameters that were obtained by minimizing $\tilde{\alpha}_\epsilon$ with indicated ϵ .

of $\tilde{\alpha}_\epsilon$ for 11 selected values of ϵ and with random starting parameters x . Next to the resulting minima for each ϵ , the number of iterations (averaged out over ten random starting points) needed to solve the respective optimization problems is listed. For small ϵ , and consequently poor smoothing, this number becomes huge. However, having the smoothing parameter at hand to tune the level of smoothing, the amount of required iterations can be drastically decreased by following a homotopy strategy, namely iteratively decreasing ϵ , and each time using the minimizer of the previous problem as the starting point. This is confirmed in Table 6.1, where the number of iterations required for this homotopy strategy and the resulting minima are listed next to the ones for which random starting points were used.

It is known that minimization of the pseudo-spectral abscissa produces a balance between asymptotic and initial decay rate for different ϵ . In particular, minimizing α_ϵ amounts to the minimal spectral abscissa for $\epsilon \rightarrow 0$, and α_ϵ minimizes the H_∞ -norm if ϵ is such that $\min_x \alpha_\epsilon = 0$, see [10]. In our case, we obtain a similar trade-off by minimizing the smoothed spectral abscissa. For ϵ going to zero, we also converge to the minimal spectral abscissa, and for a particular value of ϵ the H_2 -norm is minimized. By the relation $\alpha_{\epsilon/2} < \tilde{\alpha}_\epsilon$, it is reasonable to expect that the pseudo-spectral abscissa, evaluated in the minimizers of $\min_x \tilde{\alpha}_\epsilon(x)$, will also be pushed down when $\tilde{\alpha}_\epsilon$ is minimized for increasingly smaller ϵ . This is confirmed by the fourth column in Table 6.1.

To study the behaviour of the eigenvalues and corresponding pseudo-spectra in the minimizers of the smoothed spectral abscissa, Figure 6.3(a)-(d) depicts the pseudo-spectra at four $\tilde{\alpha}_\epsilon$ -minimizers, namely for $\epsilon = 2 \cdot 10^{-1, -3, -5, -7}$. For the first value of ϵ , both the smoothed and spectral abscissa are positive and the minimizer is not stabilizing, as seen in the spectrum plotted in Figure 6.3(a). For $\epsilon = 2 \cdot 10^{-3}$, the minimal smoothed spectral abscissa equals $-0.0270\dots$, which guarantees a stable system. As seen in Figure 6.3(b), the eigenvalues are indeed all in the left half complex plane. Since the minimum is very close to zero, we can expect $2 \cdot 10^{-3}$ to be close to the maximal ϵ for which a stabilizing solution can be found. Solving optimization problem (4.3) yields an optimal value for ϵ of $2.048 \cdot 10^{-3}$, which is indeed only slightly higher. Note that this corresponds to a minimal H_2 -norm of approximately 22. Further decrease of ϵ results in smaller and smaller minimal smoothed spectral abscissae. As observed in the two bottom frames of Figure 6.3, the rightmost eigenvalues of the optimal spectra become more and more aligned on a vertical line, indicating convergence to the typical spectrum configuration for a minimized spectral abscissa. Figures 6.3(b)-(c)-(d) thus represent three instances out of the range of stabilizing solutions that compromise between a minimal spectral abscissa on the one hand, and a minimal H_2 -norm on the other hand.

From relation (5.2), we can deduce that the H_∞ -norm equals γ^{-1} for which γ corresponds to the pseudo-spectrum Λ_γ that is exactly contained in the left half complex plane. If we have a closer look at the last three frames of Figure 6.3, we see that this is the case for the pseudo-spectra with $\gamma = 10^{-1.2}, 10^{-1.6}$ and 10^{-2} , indicating that the H_∞ -norm grows as ϵ is decreased. So, although the H_∞ -norm was not minimized here, the set of smoothed spectral abscissa minimizers appears to result in the same qualitative H_∞ behaviour as would be the case for a range of pseudo-spectral abscissa minimizations, see [9].

Let us now investigate the H_2 - and H_∞ -norms of the two sets of stabilizing minimizers, one belonging to the smoothed spectral abscissa and the other to the pseudo-spectral abscissa. We denote them as the functions $\chi_1^*(\epsilon)$ and $\chi_2^*(\epsilon)$ depending on the ϵ used in the respective minimizations. In order to be able to compare these two functions, we introduce $\epsilon_1(s)$ and $\epsilon_2(s)$ as the epsilons that yield s as minimum, i.e. such that

$$\begin{aligned} \min_x \tilde{\alpha}_{\epsilon_1(s)}(A(x)) &= s, \\ \min_x \alpha_{\epsilon_2(s)}(A(x)) &= s. \end{aligned}$$

In this way we obtain two new functions $x_1^*(s)$ and $x_2^*(s)$ as the respective minimizers of the smoothed and pseudo-spectral abscissa, with smoothing epsilons $\epsilon_1(s)$ and $\epsilon_2(s)$, and thus with minima $s \leq 0$. Concisely put,

$$\begin{aligned} x_1^*(s) &:= \chi_1^*(\epsilon_1(s)) = \arg \min_x \tilde{\alpha}_{\epsilon_1(s)}(A(x)), \\ x_2^*(s) &:= \chi_2^*(\epsilon_2(s)) = \arg \min_x \alpha_{\epsilon_2(s)}(A(x)). \end{aligned}$$

According to Remark 2 following Theorem (4.1) and equivalence (5.2), $x_1^*(s)$ and $x_2^*(s)$ respectively minimize the H_2 -norm and H_∞ -norm of a shifted system with transfer function \mathbf{H}_s . This justifies comparing x_1^* and x_2^* for the same s .

Because we are in the end only interested in the properties of the unshifted systems, we show in Figure 6.4, as a function of s , the norms $\|zI - A(x_1^*(s))\|_{\mathcal{H}_2}$ and $\|zI - A(x_2^*(s))\|_{\mathcal{H}_2}$. In other words, we compare the H_2 -norms of the *unshifted* transfer function, evaluated at the smoothed spectral abscissa minimizers $x_1^*(s)$ on the

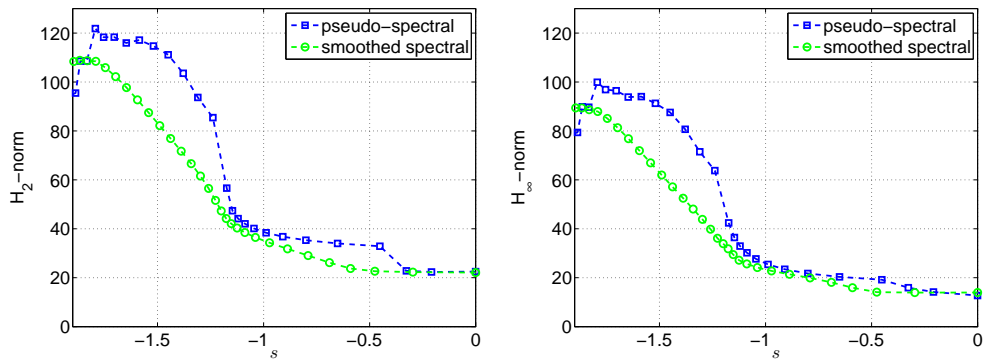


FIG. 6.4. The H_2 -norm (left) and the H_∞ -norm (right) of the unshifted systems for the minimizers $x_1^*(s)$ obtained by the minimization of the smoothed spectral abscissa (with \circ), and for the minimizers $x_2^*(s)$ obtained by the minimization of the pseudo-spectral abscissa (with \square).

one hand, and at the pseudo-spectral abscissa minimizers $x_2^*(s)$ on the other hand. In the frame of Figure 6.4, we see that the H_2 -norms of the smoothed spectral abscissa minimizers are everywhere smaller than those of the pseudo-spectral abscissa minimizers, except for s very close to the minimal spectral abscissa. For s close to zero, the difference between the two H_2 -norms becomes very small, and for $s = 0$, the H_2 -norm of the smoothed spectral abscissa minimizer is only just below the one of the pseudo-spectral abscissa minimizer. This implies that the optimal H_∞ -minimizer, being the pseudo-spectral abscissa minimizer $x_2^*(0)$, is accompanied by an H_2 -norm that is only slightly worse compared to the optimal H_2 -norm.

We now make the same comparison for the H_∞ -norm. In the right frame of Figure 6.4, we have plotted $\|zI - A(x_i^*(s))\|_{\mathcal{H}_\infty}$ for $i = 1, 2$. Again, the difference between the H_∞ -norm evaluated at the pseudo-spectral abscissa minimizers and smoothed spectral abscissa minimizers is small for s between -1 and 0 . For $s = 0$, the optimal H_∞ -norm evaluated at x_2^* is naturally smaller than the H_∞ -norm at x_1^* . Surprisingly though, for almost all of the other shifts, the H_∞ -norms of the smoothed spectral abscissa minimizers are better than the H_∞ -norms of the pseudo-spectral abscissa minimizers.

6.3. Computational cost. Finally, we compare the computational cost of the smoothed spectral abscissa $\tilde{\alpha}_\epsilon$ with two other robust stability measures: the pseudo-spectral abscissa α_ϵ and the robust spectral abscissa α_δ . Each of these measures can be used as Φ_{stab} in (1.1) and as such, will be evaluated several times in the inner iterations of an optimization algorithm. Thus, the efficiency by which Φ_{stab} can be evaluated, has a direct influence on the overall efficiency of the optimization method for solving (1.1).

The details of the numerical methods used to compute each measure are listed in Table 6.2. From these, the criss-cross algorithm with a structure preserving Hamiltonian eigenvalue solver is to be preferred. To the best of our knowledge, the listed bisection algorithm is the only known implementation for computing α_δ . Specifically, we bisection until an absolute tolerance of $10\epsilon_{\text{mach}}$ is satisfied and in each bisection step we check the feasibility of an SDP with SeDuMi 1.1R3.

Regarding the smoothed spectral abscissa, we use Dekker–Brent, implemented by MATLAB’s `fzero` with an absolute tolerance ϵ_{mach} , to find the unique root of the function $g(s) := 1/f(A, s) - \epsilon$. The reason for using the reciprocal instead of

Φ_{stab}	algorithm	convergence	inner solve (software)
α_ϵ	criss-cross [11]	quadratic	Hamiltonian (Hapack based on [5])
α_δ	bisection [8]	linear	SDP (YALMIP [22], SeDuMi [27])
$\tilde{\alpha}_\epsilon$	Dekker–Brent	superlinear	Lyapunov (Bartels–Stewart [4])

TABLE 6.2

Algorithms to compute the three stability measures.

Ex.	Φ_{stab}	ϵ, δ (logspace)	min		mean		max	
			(sec)	(its)	(sec)	(its)	(sec)	(its)
1	α_ϵ	(-15,0,20)	1.20e-03	(3)	1.79e-03	(4)	2.71e-03	(8)
	α_δ	(-2,0,20)	5.58e+00	(32)	6.18e+00	(32)	7.71e+00	(32)
	$\tilde{\alpha}_\epsilon$	(-15,0,20)	4.23e-03	(7)	7.92e-03	(16)	1.57e-02	(30)
2	α_ϵ	(-12,0,20)	2.16e-03	(3)	3.09e-03	(5)	5.49e-03	(8)
	α_δ	(-03,0,20)	2.99e+01	(54)	1.79e+02	(149)	1.63e+03	(≥ 999)
	$\tilde{\alpha}_\epsilon$	(-15,0,20)	6.31e-03	(10)	1.25e-02	(22)	2.20e-02	(39)

TABLE 6.3

Timings and inner iterations of the three stability measures. The timings were done with 100 samples and the inner iterations are given as the number of inner solves in Table 6.2.

$f(A, s) - 1/\epsilon$ is that the former is better behaved numerically. Most of the time, we observed superlinear convergence. Recall that evaluating $f(A, s)$ involves solving a Lyapunov equation, which is done by the Bartels–Stewart algorithm, implemented by `lyap` in MATLAB.

We remark that our implementation for computing $\tilde{\alpha}_\epsilon$ is very preliminary, but it seems to work well for the model problems we tried. Besides some heuristics on setting up a bracketing interval, the procedure is quite robust. As far as efficiency goes, there is a lot of room for improvement. An obvious improvement is the inner loop of `fsolve` where $f(A, s)$ is evaluated for fixed A but different shifts s . Since we solve the Lyapunov equations independently for each shift, we do not make use of the fact that we can reuse the computed Schur factorizations in Bartels–Stewart. In exact arithmetic, only one factorization would suffice. Furthermore, using Dekker–Brent to solve $g(s) = 0$ has the benefit of robustness, but we sometimes need a lot of work to find a bracketing interval. Since $f(A, s)$ is smooth and convex, a safe-guarded method based on Newton, may be more efficient. However, it is beyond the scope of the current article to implement this.

In Table 6.3 we have summarized timings for the systems that we have examined earlier with the control parameters x set to zero. However, since these three measures are quite different, comparing them is somewhat arbitrary. In order to have an impression of the computational cost, we computed each measure for a sensible range of its regularization parameter ϵ or δ . It is clear from the table that the pseudo- and smoothed spectral abscissa are comparable in computational cost, and that the robust spectral abscissa is orders of magnitudes slower.

7. Conclusions. A smooth relaxation of the non-smooth spectral abscissa function was introduced as an alternative stability measure, with the advantage that derivative based optimization techniques can readily be used for its optimization. Formulae for the efficient computation and derivative evaluation of the smoothed spectral abscissa were deduced based on the solution of a primal-dual Lyapunov equation pair.

Besides its direct minimization, which can be used to find stabilizing controllers,

a second optimization formulation was shown to be applicable to solve fixed-order H_2 -optimization problems. Moreover, a guaranteed bound on the distance to instability was established by relating the results to the H_∞ -norm. The robust stabilization by use of these two optimization problems involving the smoothed spectral abscissa was illustrated with numerical examples and also a comparative study of the computational complexity cost was made.

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