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Stability analysis of some classes of TCP/AQM networks

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The local stability analysis of some classes of nonlinear time-delay systems, encountered as fluid flow models for TCP/AQM networks, is addressed. Necessary and sufficient conditions for the asymptotic stability of the linearized models are obtained. Nonlinear stability conditions are derived using a Lyapunov-Krasovskii functional approach.

Keywords: Fluid-flow models; Nonlinear time-delay systems; Stability

1 Introduction

The analysis of fluid flow models for the behavior of high-speed networks represents a subject of recurring interest in the last years. Recently, in Misra *et al.* (2000) a fluid-flow model describing accurately the behavior of congested routers in TCP/AQM networks was introduced. A simplified version of this model was used in Hollot *et al.* (2002) to investigate the qualitative properties of TCP/AQM networks. These models express AQM schemes as classical feedback control schemes. Motivated by the feedback control interpretation, proportional and proportional-integral controllers have been proposed as AQM strategies in Hollot *et al.* (2002), and controllers based on the optimization of a $\mathcal{H}^\infty$ cost function in Quet and Özbay (2004), respectively. It was shown that such controllers improve the performance obtained with standard AQM controllers (e.g. based on Random Early Detection (RED)). The works of Hollot *et al.* (2002) and Quet and Özbay (2004) relied on a linearization around the equilibrium of the nonlinear model. Moreover, only *sufficient* stability conditions for the proposed controllers were derived.

In Hollot and Chait (2001), a simplified version of the model introduced in Hollot *et al.* (2002) was used to address the nonlinearities appearing in the TCP-fluid-flow model. A proportional packet marking function was considered as AQM strategy. For the case of a delay-free packet marking, it was proved that the equilibrium point is globally asymptotically stable for all values of the controller gain. For the case of a delayed packet marking, a Lyapunov-Razumikhin approach was used to prove *delay-independent* local asymptotic stability of the equilibrium for *sufficiently small* values of the controller gain.

In this paper we develop a thorough local stability analysis for the model considered in Hollot and Chait (2001). More explicitly, assuming a proportional delayed packet marking strategy, we first obtain *necessary and sufficient* stability conditions for the linearized model by using frequency domain tools. Also a simple to check *delay-dependent* sufficient stability condition along with robust stability conditions are provided.

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Secondly, we explicitly address the nonlinearities of the model within a Lyapunov-Krasovskii framework, where the Lyapunov-Krasovskii functional is generated from the Lyapunov-Razumikhin function candidate by adding some double integral terms. This leads to less conservative stability conditions. It is important to mention that to the best of our knowledge, there does not exist a nonlinear Lyapunov-Krasovskii based stability analysis for such a model in the literature. A simple *delay-dependent* stability condition in terms of the network parameters for the local asymptotic stability of the equilibrium is obtained. Furthermore, the local stability analysis is complemented by providing an estimate of the attraction region of the equilibrium. Note that some preliminary results in this direction have been presented by Melchor-Aguilar and Niculescu (2003). Note also that in the stability study of a related fluid flow model by Michiels and Niculescu (2005) nonlinearities are explicitly addressed too, yet in the context of investigating the occurrence of other types of attractors than equilibria (e.g. chaotic attractors).

The main goal of the paper is to illustrate the potential impact of the methods and tools for the stability analysis of time-delay systems to fluid-flow models for TCP/AQM networks. Besides this, it also illustrates how scaling properties of the model can be exploited. From the point of view of stability theory for time-delay systems a scaling of time will be used to improve conditions or to shape estimates obtained within a Lyapunov approach. Such a technique has not been adopted so far in the literature.

The remaining part of the paper is organized as follows: Section 2 introduces the mathematical model and scaling properties of the solutions. The linear stability analysis of the equilibrium is given in Section 3, and the nonlinear stability analysis in Section 4. A numerical example illustrates the main results in Section 5. Concluding remarks end the paper.

2 Mathematical Model and Scaling Properties

We consider a dynamic fluid-flow model introduced in Hollot *et al.* (2002) for describing the behavior of a TCP/AQM network which ignores the time-out and slow start mechanisms. Such a model, relating the average value of key network variables of n homogeneous TCP-controlled sources and a single congested router, is described by the following coupled nonlinear differential equations with time-varying delay

$$\begin{cases} \dot{w}(t) = \frac{1}{\tau(t)} - \frac{1}{2}\frac{w(t)w(t-\tau(t))}{\tau(t-\tau(t))}p(t-\tau(t)), \\ \dot{q}(t) \; = n(t)\frac{w(t)}{\tau(t)} - c, \end{cases} \tag{1}$$

where $w(t)$ denotes the average of TCP windows size (packets), $q(t)$ is the average queue length (packets), $\tau(t) = \frac{q(t)}{c} + \tau_p$ is the round-trip time (secs) where τ_p represents the propagation delay, c is the link capacity (packets/sec), $n(t)$ is the number of TCP sessions and $p(\cdot)$ is the probability of a packet marking. The queue length $q(t)$ and window-size $w(t)$ are positive and bounded quantities, i.e., $q(t) \in [0, q_{\max}]$ and $w(t) \in [0, w_{\max}]$. The function $p(\cdot)$ takes values only in $[0, 1]$. The first differential equation describes the TCP window control dynamics: the first term in the right-hand side corresponds to the window's additive-increase behavior, and the second term to the multiplicative-decrease behavior. The second equation describes the bottleneck queue length as the difference between the packet arrival rate $\frac{n(t)w(t)}{\tau(t)}$ and the link capacity c, assuming that there are no internal dynamics in the bottleneck.

In Hollot and Chait (2001); Hollot *et al.* (2002) it was assumed that $n(t) \equiv n$ and $\tau(t) \equiv \tau$ are constants. The assumption on the delay can be considered as a good approximation when the queueing delay is much smaller than the propagation delay, which occurs when the link capacity c is sufficiently large. As argued in Kelly (2000), routers hardware and network capacity continue to improve rapidly reducing queueing delays, whereas the propagation delays are fixed. Also in Hollot and Chait (2001) a proportional AQM scheme was taken, where $p(t) = k_p\, q(t)$, with $k_p > 0$ (see Hollot *et al.* (2002) for a discussion on the benefits of using a proportional controller). This resulted in the closed-loop system

$$\begin{aligned} \dot{w}(t) &= \frac{1}{\tau} - \frac{k_p}{2\tau} w(t) w(t-\tau) q(t-\tau), \\ \dot{q}(t) &= \frac{n}{\tau} w(t) - c, \end{aligned} \tag{2}$$

whose unique equilibrium point is given by

$$(w^*, q^*) = \left(\frac{\tau c}{n}, \frac{2n^2}{k_p(\tau c)^2} \right).$$

In Hollot *et al.* (2001) it was motivated that if $w^* \gg 1$, then the local behavior of (2) around the equilibrium can be approximated by the local behavior of

$$\begin{aligned} \dot{w}(t) &= \frac{1}{\tau} - \frac{k_p}{2\tau} w^2(t) q(t-\tau), \\ \dot{q}(t) &= \frac{n}{\tau} w(t) - c, \end{aligned} \tag{3}$$

having the same equilibrium (see the appendix for a brief mathematical justification). Notice that the condition $w^* \gg 1$ imposes a restriction on the network parameters for considering (3) as a good approximation of (2). However, in Hollot *et al.* (2001) it is argued that for a typical range of network parameters arising in practice this condition is satisfied (in Hollot and Chait (2001) and Hollot *et al.* (2001) the delay in the window dynamics is for this reason eventually neglected. In Hollot *et al.* (2002), where a linearized analysis of various types of controllers for (1) is performed, including proportional controllers, it is eventually taken out of the control loop and absorbed into a block modeling high frequency uncertainty). In the light of this fact, taking the delay in the window dynamics explicitly into account, i.e. considering the model (2) instead of model (3), would *unnecessarily* complicate the stability analysis, while the induced technicalities would distract the attention from the main goals of the paper.

In order to simplify our subsequent analysis we introduce the following *scaling* of state and time:

$$t^{(\text{new})} = \frac{t^{(\text{old})}}{\tau}, \quad \widetilde{w} = w, \quad \widetilde{q} = \frac{q}{n}. \tag{4}$$

Then the system (3) is transformed to

$$\begin{aligned} \dot{\tilde{w}}(t) &= 1 - \frac{K}{2} \tilde{w}^2(t) \tilde{q}(t-1), \\ \dot{\tilde{q}}(t) &= \tilde{w}(t) - C, \end{aligned} \tag{5}$$

where

$$C = \frac{\tau c}{n}, \text{ and } K = k_p n. \tag{6}$$

In the new coordinates the equilibrium is given by

$$(\widetilde{w}^*, \widetilde{q}^*) = \left(C, \frac{2}{KC^2} \right).$$

The model representation in the new coordinates only depends on *two* parameters (K, C) instead of the *four* original parameters (k_p, n, c, τ). This facilitates the study of the dependence of the dynamic behavior of the controlled network on its parameters. It will also allow to display stability regions in the whole parameter space with only one 2D-plot.

3 Linear Stability Analysis

When linearizing (5) around to the equilibrium point, we arrive at the following second-order delay differential equation:

$$\ddot{y}(t) + \frac{2}{C}\dot{y}(t) + \frac{KC^2}{2}y(t-1) = 0, \tag{7}$$

where $y(t) = \tilde{q}(t) - \tilde{q}^*$. It is well known that (7) is asymptotically stable if and only if the characteristic equation

$$s^2 + \frac{2}{C}s + \frac{KC^2}{2}e^{-s} = 0 \tag{8}$$

has no roots with nonnegative real parts Hale and Verduyn-Lunel (1993); Niculescu (2001); Gu *et al.* (2003).

Necessary and sufficient stability conditions are expressed in the following theorem:

THEOREM 3.1 *Equation (7) is asymptotically stable if and only if the positive coefficients K and C belong to the stability region plotted in Fig. 1, whose boundary in the parameters space (K, C) is described by*

$$K = \frac{\omega^4 \sin^2(\omega)}{2\cos^3(\omega)}, \; C = \frac{2\cos(\omega)}{\omega \sin(\omega)}, \; \textit{where } \omega \in \left(0, \frac{\pi}{2}\right). \tag{9}$$

Proof First, observe that, since $K > 0$ and $C > 0$, $s = 0$ is not a root of (8). Suppose that (8) has a pure imaginary root $s = i\omega \neq 0$. Then, a direct calculation yields

$$K = \frac{\omega^4 \sin^2(\omega)}{2\cos^3(\omega)} \text{ and } C = \frac{2\cos(\omega)}{\omega \sin(\omega)}. \tag{10}$$

This parametrization defines a countable number of curves in the parameter space (K, C) with $K > 0$ and $C > 0$, and each one of them is obtained by varying ω in an interval $(j\pi, (j + 1/2)\,\pi)$, $j = 0, 2, 4, ...$ These curves partition the plane (K, C) into a set of connected domains.

From the argument principle is easy to show that for all (K, C) values inside the open domain, bounded by the curve for $j = 0$ and the coordinate axes, equation (8) has no roots with strictly positive real part. □

Remark 1 As an alternative to applying the argument principle, the stability region in the (K, C) plane can also be selected from the partition defined by the curves (10), by characterizing the *crossing directions* of eigenvalues along these curves as the parameters are varied. See Section 2 of Michiels and Niculescu (2005) where such an approach is applied to (2).

The following corollary addresses the robustness of stability of the equilibrium point w.r.t. changes in the network parameters:

COROLLARY 3.2 *Assume that a controller with gain k_p stabilizes a network with parameters (n_0, τ_0, c_0). Then it stabilizes a network with parameter (n, τ, c), where*

$$n \geq n_0, \; c \leq c_0 \textit{ and } \tau \leq \tau_0.$$

Proof From (6) a decrease of τ and c implies a decrease of C for constant K, hence it is clear that the stability region cannot be left. So we can restrict ourselves to studying the effect of increasing n.

Let $(K_0, C_0) = (k_p n_0, \tau_0 c_0/n_0)$ and $(K, C) = (k_p n, \tau_0 c_0/n)$. Then we have $CK = C_0 K_0$, or

$$(\log C - \log C_0) = -(\log K - \log K_0). \tag{11}$$

In other words, an increase of n corresponds in the $(\log K, \log C)$ plane to a shift downwards along a line *with slope minus one.*

Next, we look at the slope of the stability boundary (9). The expression for K implies the existence of a function $K \in (0,\ +\infty) \to \omega(K) \in (0,\ \pi/2)$, satisfying $\omega'(K) \geq 0$. Furthermore, we have from (9):

$$KC^3 = 4\frac{\omega(K)}{\sin(\omega(K))}$$

or

$$\log C = -\frac{1}{3}\log K + \frac{1}{3}\log\frac{4\omega(K)}{\sin(\omega(K))}.$$

Since $\omega'(K) \geq 0$ and the function $\omega \in (0,\ \pi/2) \to \log(4\omega/\sin\omega)$ is increasing, the derivative of the second term of the right-hand side w.r.t. K, and hence w.r.t. $\log K$, is positive. So we have along the stability boundary:

$$\frac{d(\log C)}{d(\log K)} \geq -\frac{1}{3}. \tag{12}$$

From (11) and (12) it follows that the stability region cannot be left by increasing n. This completes the proof, whose main ideas are illustrated in Figure 1 (right). □

Remark 2 Corollary 3.2 is similar to Proposition 2 in Hollot *et al.* (2002), which states that designing the controller for the largest expected values of τ and c, and the smallest expected value of n yields a robustly stabilizing controller. The main difference with Hollot *et al.* (2002) lies in the fact that with our approach the *controller's gain* can be *selected* based on the *exact stability region, not on an estimate of this region.*

The following result provides an easy-to-check sufficient stability condition for equation (7):

Proposition 3.3 *Equation (7) is asymptotically stable if*

$$0 < K < \frac{4}{C^3} \text{ or } 0 < k_p < \frac{4n^2}{(\tau c)^3}. \tag{13}$$

Proof First, observe that for any given (K_0, C_0), $K_0 > 0$, belonging to the boundary of the stability region, there is a unique $\omega_0 \in \left(0, \frac{\pi}{2}\right)$ such that $K_0 = \frac{1}{{C_0}^3}\frac{4\omega_0}{\sin(\omega_0)}$. Thus, given $C_0 > 0$, a necessary and sufficient condition for stability of (7) is

$$0 < K < \frac{4\omega_0}{\sin(\omega_0)}\frac{1}{{C_0}^3}.$$

Observing that for all $\omega_0 \in \left(0, \frac{\pi}{2}\right)$:

$$4 < \frac{4\omega_0}{\sin(\omega_0)},$$

we arrive at the desired sufficient stability condition. □

From the parametrization (9) it follows that, when $w^*(=\widetilde{w}^*=C)$ is large, the corresponding values for ω are close to zero, and (9) can be approximated with

$$K \approx \frac{\omega^6}{2} \text{ and } C \approx \frac{2}{\omega^2}.$$

It follows that $K \approx 4/C^3$, and therefore the conservatism of the sufficient condition (13) vanishes as C tends to infinity. These results are illustrated on Figure 1.

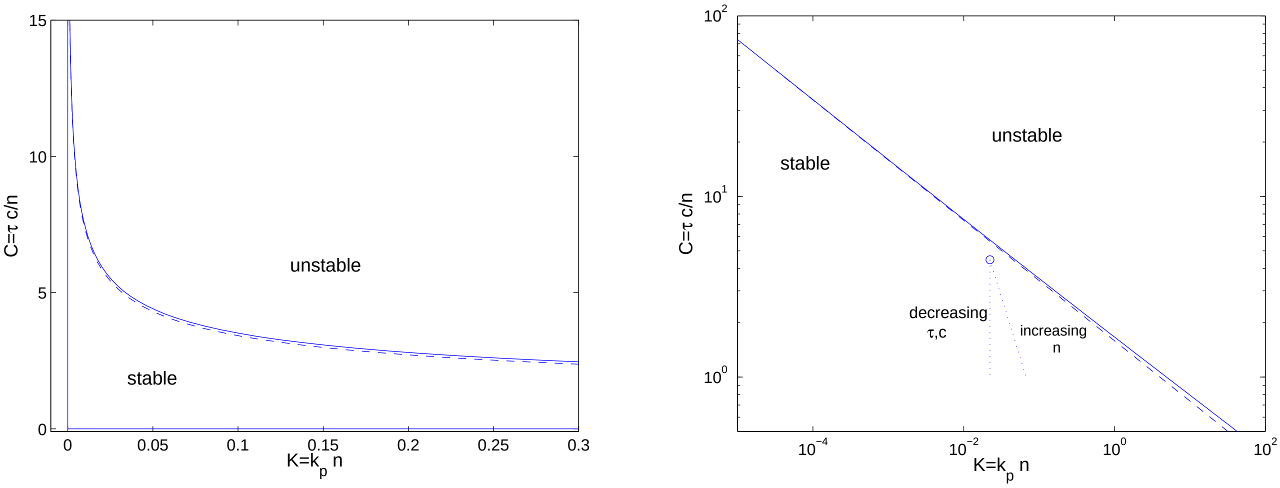

Figure 1. Stability region of (7) in the (K,C) plane using a linear scale (left) and a logarithmic scale (right). The solid line bounds the stability region of (7), the dashed line corresponds to the sufficient condition (13). In the logarithmic plot, the effect of decreasing τ, c and increasing n on a workpoint ('o') is also illustrated.

4 Nonlinear stability analysis

From (5) one obtains the following second order delay differential equation:

$$\ddot{y}(t) + K\left(0.5\dot{y}^2(t) + C\dot{y}(t)\right)\left(y(t-1) + \tilde{q}^*\right) + \frac{KC^2}{2}y(t-1) = 0. \tag{14}$$

In order to define a particular forward solution of (14), one has to know an initial function $\varphi(\theta)\,,\ \theta \in [-1,0]$. We assume that $\varphi \in \mathcal{C}^1[-1,0]$, i.e., φ is a real continuous function having a continuous derivative in $(-1,0)$, a right-hand continuous derivative at -1 and a left-hand continuous derivative at 0. Then the vector $(\varphi,\dot{\varphi})$ belongs to $\mathcal{C}\left([-1,0]\,,\mathbb{R}^2\right)$, the Banach space of continuous functions mapping $[-1,0]$ into $\mathbb{R}^2$ and equipped with the supremum norm, $\|(\varphi,\dot{\varphi})\| = \sup_{\theta\in[-1,0]}\|(\varphi(\theta),\dot{\varphi}(\theta))\|$.

For any φ, satisfying the above constraints, we denote by $t \geq -1 \to y(\varphi)(t) \in \mathbb{R}$ the unique forward continuous solution of (14), satisfying $y(\varphi)(\theta) = \varphi(\theta),\ \theta \in [-1,0]$. The function segment $\theta \in [-1,0] \to y(\varphi)(t+\theta)$ is denoted by $y_t(\varphi)$. For simplicity of the notation, we will in the sequel often omit the explicit dependence of a solution on the initial condition and write $y(t)$ resp. y_t, instead of $y(\varphi)(t)$ resp. $y_t(\varphi)$.

We consider the following Lyapunov-Krasovskii functional:

$$v\left(\varphi,\dot{\varphi}\right) = \frac{KC^2}{2}\varphi(0)^2 + \dot{\varphi}^2(0) + \frac{KC^2}{2}\int_{-1}^{0}\int_{\theta}^{0}\dot{\varphi}^2(\xi)d\xi d\theta. \tag{15}$$

Remark 1 There exist constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\gamma_1 \left\|(\varphi(0), \dot{\varphi}(0))\right\|^2 \leq v\left(\varphi, \dot{\varphi}\right) \leq \gamma_2 \left\|(\varphi, \dot{\varphi})\right\|^2. \tag{16}$$

For example, condition (16) is fulfilled with

$$\gamma_1 = \min\left\{\frac{KC^2}{2}, 1\right\}, \quad \gamma_2 = \max\left\{\frac{KC^2}{2}, 1 + \frac{KC^2}{4}\right\}. \tag{17}$$

Taking the time-derivative of the functional (15) along a trajectory of (14) yields

$$\frac{d}{dt} v\left(y_t, \dot{y}_t\right) = KC^2 \dot{y}(t)(y(t) - y(t-1)) + \frac{KC^2}{2}\dot{y}^2(t) - \frac{KC^2}{2}\int_{-1}^{0} \dot{y}^2(t+\theta) d\theta$$
$$-2K\dot{y}(t)\left(0.5\dot{y}^2(t) + C\dot{y}(t)\right)\ \left(y(t-1) + \tilde{q}^*\right),$$

and taking into account that

$$KC^2\dot{y}(t)\ (y(t) - y(t-1)) = KC^2\dot{y}(t)\int_{-1}^{0} \dot{y}(t+\theta)d\theta \leq \frac{KC^2}{2}\left(\dot{y}^2(t) + \int_{-1}^{0}\dot{y}^2(t+\theta)d\theta\right),$$

we get

$$\frac{d}{dt} v\left(y_t, \dot{y}_t\right) \leq KC^2\dot{y}^2(t) - 2K\dot{y}(t)\left(0.5\dot{y}^2(t) + C\dot{y}(t)\right)\left(y(t-1) + \tilde{q}^*\right).$$

Next, we assume that

$$\tilde{q}^* - C > 0 \tag{18}$$

and define

$$\mathcal{U} := \left\{(\varphi, \dot{\varphi}) : v\left(\varphi, \dot{\varphi}\right) < \gamma_1\left(\tilde{q}^* - C\right)^2 \text{ and } \left\|(\varphi, \dot{\varphi})\right\| < \tilde{q}^* - C\right\}. \tag{19}$$

When $(y_t, \dot{y}_t) \in \mathcal{U}$ for some $t \geq 0$, we have $y(t-1) + \tilde{q}^* - C > 0$. Using this property and $\dot{y}(t) = \dot{\tilde{q}}(t) = \tilde{w}(t) - C$, implying $\dot{y}(t) \geq -C$ and

$$-2K\dot{y}(t)\left(0.5\dot{y}^2(t) + C\dot{y}(t)\right) \leq -KC\dot{y}^2(t),$$

we arrive at

$$\frac{dv(y_t, \dot{y}_t)}{dt} \leq -w(y_t, \dot{y}_t) \leq 0, \tag{20}$$

where

$$w(y_t, \dot{y}_t) = KC\dot{y}^2(t)(y(t-1) + \tilde{q}^* - C).$$

Furthermore, the inequality

$$\gamma_1 \left\|(y(t), \dot{y}(t))\right\|^2 \leq v\left(y_t, \dot{y}_t\right) < \gamma_1\left(\tilde{q}^* - C\right)^2$$

implies

$$\|(y(t), \dot{y}(t))\| < \tilde{q}^* - C. \tag{21}$$

From (19)-(21) and the continuity of the solutions one can conclude that under the assumption (18):

(i) $\mathcal{U}$ is a positively invariant set with respect to equation (14);
(ii) v is a Lyapunov functional on $\mathcal{U}$.

For an initial condition $(\varphi, \dot{\varphi}) \in \mathcal{U}$ we now show that the corresponding solution $t \geq 0 \to y(\varphi)(t)$ asymptotically converges to zero. Inequality (20) implies that the function $t \geq 0 \to v(y_t, \dot{y}_t)$ is nonincreasing, and thus

$$\gamma_1 \left(y^2(t) + \dot{y}^2(t)\right) \leq v(y_t, \dot{y}_t) \leq v(\varphi, \dot{\varphi}), \ \forall t \geq 0.$$

So y and $\dot{y}$ are uniformly bounded. It follows that $\ddot{y}$ is uniformly bounded, implying that the function $t \geq 0 \to w\,(y_t, \dot{y}_t)$ is uniformly continuous. From (20) we also have

$$\int_0^t w\,(y_\xi, \dot{y}_\xi)\, d\xi \leq v(\varphi, \dot{\varphi}) - v(y_t, \dot{y}_t).$$

Since $v(y_t, \dot{y}_t)$ is nonincreasing and bounded from below by zero, it converges as $t \to \infty$, and hence

$$\lim_{t \to \infty} \int_0^t w\,(y_\xi, \dot{y}_\xi)\, d\xi$$

exists and is finite. An Application of Barbalat's Lemma (see Barbalat (1959)) yields

$$\lim_{t \to \infty} w\,(y_t, \dot{y}_t) = 0. \tag{22}$$

The uniform continuity of the function $t \geq 0 \to w\,(y_t, \dot{y}_t)$ and (22) imply that $(y_t, \dot{y}_t)$ converges to the largest invariant set of (14) where $w \equiv 0$, being the zero solution. Hence, the set $\mathcal{U}$ is an estimate of the region of attraction.

Note that, by combining (19) with the upper bound on $v\,(\varphi, \dot{\varphi})$ in (16) and using $\gamma_1 \leq \gamma_2$, one obtains a more conservative, yet computational more tractable estimate for the attraction region:

$$\mathcal{V} = \left\{(\varphi, \dot{\varphi}) : \|(\varphi, \dot{\varphi})\| < \sqrt{\frac{\gamma_1}{\gamma_2}}(\tilde{q}^* - C)\right\} \subseteq \mathcal{U}. \tag{23}$$

Summarizing the above results we arrive at the following result which improves the stability condition described in Hollot and Chait (2001):

Theorem 4.1 *The zero solution of equation (14) is locally asymptotically stable if $\tilde{q}^* - C > 0$, being equivalent with*

$$0 < K < \frac{2}{C^3} \text{ and } 0 < k_p < \frac{2n^2}{(\tau c)^3}. \tag{24}$$

The region of attraction includes the sets $\mathcal{U}$ and $\mathcal{V}$, defined in (19) and (23).

Note that the right-hand sides of the sufficient (linear) stability condition (13) and the sufficient (nonlinear) stability condition (24) differ with a *factor two*. On the other hand, with condition (24) the nonlinearities of the system are considered and estimates for the region of attraction are given.

Remark 2 Assume that the network parameters (n, τ, c) are constants satisfying

$$n \in [n_1, n_2], \ c \in [c_1, c_2] \text{ and } \tau \in [\tau_1, \tau_2], \tag{25}$$

and that condition (24) holds for (n_1, τ_2, c_2). Then Theorem 4.1 holds for all parameter values satisfying (25). Thus, one arrive at the result that designing the controller for the largest expected values of τ and c, and the smallest expected value of n yields a robustly stabilizing controller.

Remark 3 A functional of the form (15), or in its more general matrix form, has been applied so far to the stability analysis of linear time-delay systems under appropriate model transformations, see Gu and Niculescu (2000); Kharitonov and Melchor-Aguilar (2000); Niculescu and Annaswamy (2000). The above analysis demonstrates its potential to analyze nonlinear systems.

Remark 4 (Shaping of the attraction region) Consider the estimate of the region of attraction (23), with γ_1 and γ_2 computed from (17). We now outline an approach to modify the shape of this set, as well as the conservatism of the estimate. When applying a time-transformation $\hat{t} = Rt, \ R > 0$, to equation (14), repeating the above analysis with the functional

$$\hat{v}(\varphi, \dot{\varphi}) = \frac{KC^2}{2R^2}\varphi(0)^2 + \dot{\varphi}^2(0) + \frac{KC^2}{2R^2}\int_{-R}^{0}\int_{\theta}^{0}\dot{\varphi}^2(\xi)d\xi d\theta, \ \varphi \in \mathcal{C}^1([-R, \ 0], \ \mathbb{R}),$$

and doing the back-transformation of time, one arrives at the following estimate of the attraction domain for the zero solution of equation (14):

$$\hat{\mathcal{V}}(R) = \left\{(\varphi, \dot{\varphi}) : \left\|\left(\varphi, \frac{\dot{\varphi}}{R}\right)\right\| < \sqrt{\frac{\hat{\gamma}_1(R)}{\hat{\gamma}_2(R)}}(\tilde{q}^* - C)\right\} \tag{26}$$

with

$$\hat{\gamma}_1(R) = \min\left\{\frac{KC^2}{2R^2}, 1\right\}, \quad \hat{\gamma}_2(R) = \max\left\{\frac{KC^2}{2R^2}, 1 + \frac{KC^2}{4}\right\}.$$

When comparing (23) and (17) with (26) we have $\hat{\mathcal{V}}(1) = \mathcal{V}$ and it becomes clear that R is an additional parameter which can be used to shape or optimize the estimate of the region of attraction.

5 Example

Consider the network parameters from Hollot *et al.* (2002): $c = 3750$ packets/s, $n = 60$ flows and $\tau = 0.246 \ s$.

Then $C = 15.3750$. From Theorem 1 we have that if $K < 1.125 \times 10^{-3}$ then equation (7) is asymptotically stable. Hence, selecting $k_p < 1.8750 \times 10^{-5}$ we arrive at a stabilizing proportional controller. From the sufficient stability condition (13) one obtains $k_p < 1.83 \times 10^{-5}$.

We now consider the nonlinear stability conditions. From (24) we have that if $k_p < 9.1714 \times 10^{-6}$ then the zero solution of (14) is locally asymptotically stable. Taking $k_p = 9 \times 10^{-6}$, we get the equilibrium point $\tilde{q}^* = 15.6677$, $\tilde{w}^* = C = 15.3750$. With γ_1 and γ_2 computed from (17), resulting in $\gamma_1 = 0.0638, \ \gamma_2 = 1.0319$, the estimates of the region of attraction of the zero solution of (14), defined by (19) and (23) are given by:

$$\mathcal{U} = \{(\varphi, \dot{\varphi}) : v(\varphi, \dot{\varphi}) < 0.0857 \text{ and } \|(\varphi, \dot{\varphi})\| < 0.2927\}$$

and

$$\mathcal{V} = \{(\varphi, \dot{\varphi}) : \|(\varphi, \dot{\varphi})\| < 0.07281\}. \tag{27}$$

From (26) we have with the choices $R = \sqrt{K/2}\, C \approx 0.2526$ (which *maximizes* the ratio $\hat{\gamma}_1(R)/\hat{\gamma}_2(R)$) and $R = \tau = 0.246$:

$$\begin{aligned} \hat{\mathcal{V}}(0.2526) &= \left\{(\varphi, \dot{\varphi}) : \left\|\left(\varphi, \tfrac{\dot{\varphi}}{0.2526}\right)\right\| < 0.2882\right\}, \\ \hat{\mathcal{V}}(0.246) &= \left\{(\varphi, \dot{\varphi}) : \left\|\left(\varphi, \tfrac{\dot{\varphi}}{0.246}\right)\right\| < 0.2850\right\}. \end{aligned} \tag{28}$$

Next, we return to the *original* coordinates and time-scale (cf. Section 2). Then the equilibrium is given by $(w^*, q^*) = (15.375, 940.06)$. With $z(t) = q(t) - q^*$, the sets (27)-(28), which are all contained in the region of attraction, become:

$$\begin{aligned} \mathcal{V} = \hat{\mathcal{V}}(1) &= \{(z_t,\ \dot{z}_t) : \|(z_t,\ 0.246\ \dot{z}_t)\| < 4.368\}, \\ \hat{\mathcal{V}}(0.2526) &= \{(z_t,\ \dot{z}_t) : \|(z_t,\ 0.9737\ \dot{z}_t)\| < 17.29\}, \\ \hat{\mathcal{V}}(0.246) &= \{(z_t,\ \dot{z}_t) : \|(z_t,\ \dot{z}_t)\| < 17.10\}. \end{aligned}$$

6 Conclusions

We performed a local stability analysis of some classes of TCP/AQM networks based on fluid-flow models. We considered a packet marking strategy where the probability of a packet marking is proportional to the buffer queue length. Necessary and sufficient stability conditions for the linearized model were obtained. This allows to select the controller's gain based on the exact stability region and not on an estimate of this region, the latter occurring in the existing work. A nonlinear Lyapunov-Krasovskii based stability analysis of the equilibrium was performed and sufficient stability conditions were derived. These conditions are *delay-dependent* and improve the stability conditions described in Hollot and Chait (2001). An estimate of the attraction region along with an approach to shape or optimize this region complemented the local stability analysis of the equilibrium. Due to the fact that the scaling properties of the model were fully exploited, which allowed to reduce the number of parameters from four to two, the selection of parameters and the study of the parametric dependence of stability properties were facilitated.

In doing the analysis, we illustrated the capabilities of tools and methods for the stability analysis of time-delay systems towards nonlinear fluid-flow models for the behavior of some classes of TCP/AQM networks. In particular, we illustrated the potential of the Lyapunov Krasovskii approach, which was combined with Barbalat's Lemma to obtain stability conditions for a second order nonlinear model with time-delays. Also the importance of exploiting scaling properties was illustrated: first, to reduce the number of parameters, and, second, to shape and optimize estimates of the region of attraction of an equilibrium.

Finally, there are several directions in which this work can be extended. Concerning the study of TCP/AQM networks, the tools adopted in the paper can be applied to investigate other marking strategies as proposed by Hollot *et al.* (2002), such as proportional-integral schemes. Concerning the stability of time-delay systems in general, the investigation of the potential of a scaling of time to improve or shape conditions and to reduce conservatism within a Lyapunov approach deserves further attention.

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Appendix A: Mathematical Justification of Model (3)

In this section we present a mathematical justification of considering (3) as an approximation of (2) based on Hollot *et al.* (2001). The linearization of (2) around the equilibrium (w^*, q^*) is given by

$$\begin{aligned} \delta\dot{w}(t) &= -\frac{n}{\tau^2 c}\left[\delta w(t) + \delta w(t-\tau)\right] - \frac{k_p \tau c^2}{2n^2}\,\delta q(t-\tau), \\ \delta\dot{q}(t) &= \frac{n}{\tau}\,\delta w(t). \end{aligned} \tag{A1}$$

The characteristic function of this system is

$$f(s) = s\left(s + \alpha + \alpha e^{-\tau s}\right) + \beta e^{-\tau s},$$

where

$$\alpha = \frac{1}{\tau w^*}, \quad \beta = \frac{k_p c^2}{2n}.$$

Since $\alpha > 0$ is not a zero of $f(s)$ we can write

$$f(s) = s\,(s+\alpha)\,g(s) + \beta e^{-\tau s},$$

where

$$g(s) = 1 + \frac{1}{1+s/\alpha} e^{-\tau s}.$$

The bandwidth of the low-pass filter $\frac{1}{1+s/\alpha}$ is given by α. Therefore, if $\alpha\tau \ll 1$ we can approximate $g(s) \approx 1 + \frac{1}{1+s/\alpha}$ in small strips around the imaginary axis, which results in

$$f(s) \approx s\,(s+2\alpha) + \beta e^{-s\tau}.$$

Hence, when $\alpha\tau \ll 1$, or equivalently, $w^* \gg 1$, the linear system (A1) can from a stability analysis point of view be approximated by the following linear system

$$\begin{aligned} \delta\dot{w}(t) &= -\tfrac{2n}{\tau^2 c}\,\delta w(t) - \tfrac{k_p \tau c^2}{2n^2}\,\delta q(t-\tau), \\ \delta\dot{q}(t) &= \tfrac{n}{\tau}\,\delta w(t). \end{aligned}$$

The linear system (A2) is the linearization of the nonlinear system (3) around the equilibrium (w^*, q^*). Thus, it follows that, when $w^* \gg 1$, the local behavior of (2) can be approximated by the local behavior of (3) around the equilibrium point.