Stabilizing a Chain of Integrators Using Multiple Delays

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***Abstract*—This note addresses the output feedback stabilization problem of a chain of integrators using multiple delays. We shall prove that either n distinct delays or a proportional+delay compensator with $n-1$ distinct delays are sufficient to stabilize a chain including n integrators. We present two different approaches. Both are constructive and rely on frequency-domain techniques: on a derivative feedback approximation idea, and a pole placement idea, respectively. An illustrative example (triple integrator) is presented.**

***Index Terms*—Delay, integrator, interpolation, pole placement.**

I. Introduction

In the sequel, we address the following problem.

Problem 1: Find (necessary and/or sufficient) conditions on the $(2m+1)$-tuple (m, k_i, τ_i), $i = \overline{1, m}$ such that the (output feedback) control law defined by the chain of m distinct delay blocks (k_i, τ_i)

$$u(t) = -\sum_{i=1}^{m} k_i y(t - \tau_i) \tag{1}$$

asymptotically stabilizes the chain of n integrators: $H_{yu}(\lambda) = 1/\lambda^n$.

When one of the delays is equal to zero, we call (1) a *proportional+delay compensator*.

The main interest in analyzing control laws of the form (1) lies in the simplicity of the controller, as well as in its practical implementation facility.[1] This problem is strongly related to the increasing interest of controlling congestion phenomena in high-speed networks using a fluid approximation [16]. Indeed, delay phenomena always appear in data transfer for a given (source, destination) pair through some network channels, and, furthermore, as seen in [8], the integrator is the easiest way to represent a bottleneck in the path between the corresponding source and destination.

Some simple computations prove that the single integrator can be easily stabilized by a single delay (see, for instance, [13] and the references therein). Indeed, a positive gain guarantees the closed-loop stability of the system free of delay, and, by continuity, there exists a (sufficiently small) delay in the output such that the closed-loop stability is still preserved. The situation is completely different for a double integrator: one delay is not sufficient to stabilize it, since the system free of delay is an oscillator, and it becomes unstable for any positive and sufficiently small delay. However, the use of two appropriate delays (not necessarily rationally independent) can ensure the closed-loop stability [3] (see also [14] for a different stability argument). The result is still true if one of the delays is equal to 0 (see, e.g., [1] and [13]).

In this note, the ideas previously mentioned are generalized to the chain including n integrators. First we devote a brief discussion to the number of delay blocks, *necessary* for the stabilization of the chain. We develop arguments, which lead us to a conjecture, stating that at least $m = n$ terms in the control law (1) are necessary for stabilization. Then, we will prove that either n distinct delays in the control law or a proportional+delay compensator with $n-1$ distinct delays are *sufficient*. We use two different approaches, which are both constructive. The first approach is based on a derivative feedback idea. More precisely, starting from a stabilizing control law without delays but using output derivatives, we show that stability is preserved when the derivatives are closely approximated with (past) measurements of the output. Although the proposed approach is similar to the one proposed by Kokame and Mori in [7] (see also the references therein) for stabilizing systems without delays, the technique used here is slightly different using an interpolation based method. The second approach is based on the pole placement of the n rightmost roots of the corresponding characteristic equation using low-gain control laws, inspired by the recent work on constrained linear control of time-delay systems [9], [10].

Although the problem of stabilizing a chain of integrators using bounded input with [9], [10] or without [19] delays was already treated in the literature, the problem stated above was never considered in the multiple delays framework, where the delays are seen and interpreted as *design parameters*. Some particular cases (single integrator, double integrator with one or two delays) have already been considered [3], [10], [13], [14], but without any attempt to work out the general case.

It is important to note that the use of delays as control parameters is not new. In this sense, see the work of Pyragas [15] on stabilizing unstable periodic orbits in chaotic systems by inducing delays in the control law (the delay chosen is equal to the corresponding period), or the work of Yamanaka and Shimemura [20], where a controller of the form (1) was already encountered, but for a different control problem. More precisely, [20] proved that the minimum variance (minimized L_2-norm) of some internal model control (IMC) scheme can be made arbitrarily small for a minimum-phase plant by increasing the number of delays in the control law.

The note is organized as follows. Iin Section II, some properties (scaling, stabilizability) of (delayed) output feedback are outlined, Section III is devoted to the main results, as well as to various discussions and interpretations. Note, however, that we are not interested in discussing other performances or limitations of the control scheme, excepting the asymptotic stability in closed-loop. An illustrative example (triple integrator) is presented in Section IV, and some concluding remarks end the note. The notations are standard.

II. Properties

We discuss some theoretical properties of control laws of the form (1).

A. Scaling

A useful result in the rest of the note is the following scaling property, which indicates a natural tradeoff between "gain" and "delay" in the controller construction.

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[1]Furthermore, if $n > 1$, it is well known that there does not exist any (static output feedback) stabilizing control law (free of delays) of the form $u(t) = -ky(t)$.

Property 1: The control law

$$u(t) = -\sum_{j=1}^{m} k_j y(t - \tau_j) \tag{2}$$

is asymptotically stabilizing if and only if

$$u(t) = -\sum_{j=1}^{m} \frac{k_j}{\rho^n} y(t - \rho\tau_j), \qquad \rho > 0 \tag{3}$$

is asymptotically stabilizing.

Proof: The transformation from (2) to (3) simply involves a scaling of the closed-loop eigenvalues by $1/\rho$. □

Note that an analogous scaling property was the basis for the construction of state feedback controllers in the presence of input constraints in [10] and also played a crucial role in the study of the so-called peaking phenomena, see, e.g., [17] and [18] (and the references therein).

B. Stabilizability

An important issue in studying the stabilizability with control laws of the form (1) concerns the minimal number of output measurements, needed for the construction of a stabilizing feedback law, m. In the next section we show that $m = n$ is sufficient. The question rises whether a control law with $m < n$ output measurements could also work. For the special cases of the single and the double integrator, which have already been treated in the literature, the *answer* is *negative*, as discussed in the Introduction. Although a complete proof is missing, we believe that this result can be generalized to the following.

Conjecture 1: A chain of n integrators $(n \geq 2)$ can neither be stabilized with a chain of less than n delay blocks, nor with a proportional+delay compensator with less than $n-1$ delays. The fact that the uncontrolled system has n roots in the closed right half plane could be a motivation for the need of n controller parameters. On one hand, this argument is not strong because the conjecture concerns a *stabilizability* problem, rather than a *controllability* problem of the n rightmost eigenvalues, but on the other hand, in the state feedback case and without delay, also n controller parameters are necessary for asymptotic stabilization. A much stronger argument will be developed as a side-result in the next section (Remark 2).

We conclude with the special case of a triple integrator with a $P+$ delay controller.

Proposition 1: The chain of three integrators can never by stabilized by a proportional+delay controller, including only one delay block.

Proof: Assume that the controller has the form: $u(t) = -k_2 y(t) + k_1 y(t-\tau)$. Without any loss of generality, assume $k_i > 0$, $i = 1, 2$. Then, the corresponding characteristic equation becomes

$$\lambda^3 + k_2 - k_1 e^{-\lambda\tau} = 0. \tag{4}$$

The roots on the imaginary axis $\pm j\omega$, $\omega > 0$, if any, should satisfy the constraints:

$$\begin{cases} k_2 - k_1 \cos(\omega\tau) = 0 \\ \omega^3 - k_1 \sin(\omega\tau) = 0. \end{cases} \tag{5}$$

It is easy to see that if $k_2 > k_1$, then (5) will never have solutions. Thus, the considered controller $u(t) = -k_2 y(t) + k_1 y(t-\tau)$ does not change the system's behavior for any positive delay value, that means the *delay-independent instability* of the closed-loop system.

Assume that $k_1 > k_2$. Then, (5) has the solution $j\widetilde{\omega} = j(k_1^2 - k_2^2)^{1/6}$ on the imaginary axis, and the corresponding delay value is

$$\widetilde{\tau} = \frac{1}{(k_1^2 - k_2^2)^{\frac{1}{6}}} \arcsin\left(\sqrt{1 - \frac{k_2^2}{k_1^2}}\right). \tag{6}$$

Assume τ a free parameter of the system, and we shall analyze the behavior of the roots with respect to τ. In this sense, we need to compute $d\lambda/d\tau$ in (4), which leads to

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{3\lambda}{k_2 + \lambda^3} - \frac{\tau}{\lambda}. \tag{7}$$

If we evaluate this derivative on the imaginary axis, and we take only the real part, one gets

$$\mathbf{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}_{\mathrm{s=j}\omega} = +\frac{3\omega^4}{\mathrm{k}_1^2} \tag{8}$$

quantity which is always positive. Based on [4], [13], the *sign* of (8) gives the crossing sense of the roots: positive corresponds to instability crossing, and negative to stability, respectively.

Note that the same conclusion holds if we assume that k_1 is negative and/or if k_2 is negative. □

III. Stabilization Approaches

In this section, we show that the multiple integrator can always be stabilized with a chain of n delay blocks or with a proportional+delay controller with $(n-1)$ delays. Therefore, we explicitly construct stabilizing control laws, using two different approaches. The first approach, inspired by [7], consists of approximating output derivatives with (delayed) output measurements. This initially leads to control laws with small delays, but by Property 1 control laws with arbitrary delays can be constructed also (with a "tradeoff" on the "distance" to the imaginary axis). The second approach, inspired by [9] and [10], consists of a placement of the n dominant closed-loop eigenvalues by means of low-gain control laws.

A. Control Laws Based on Numerical Differentiation With Backward Differences

The system $y^{(n)}(t) = u(t)$ can be stabilized with the feedback law

$$u(t) = -q_0\, y(t) - q_1\, y'(t) - \ldots - q_{n-1}\, y^{(n-1)}(t) \tag{9}$$

where the polynomial $q(\lambda) = \lambda^n + \sum_{k=0}^{n-1} q_k \lambda^k$ is Hurwitz. The latter implies that $q_k > 0, \forall k = \overline{1,n}$. Hence, all the derivatives of the output, up to order $(n-1)$, are needed in the control law.

The key idea in the controller construction consists of approximating the output derivatives in (9) with (delayed) output measurements. For instance, we have

$$y'(t) \approx \frac{y(t) - y(t-\epsilon)}{\epsilon} \tag{10}$$

for small ϵ, which corresponds to an approximation

$$\lambda \approx \frac{1 - e^{-\lambda\epsilon}}{\epsilon}$$

in the frequency domain. Note that the right-hand side of (10) is the derivative of the linear approximation of y through the points $(t, y(t))$ and $(t-\epsilon, y(t-\epsilon))$. We now outline how this idea can be generalized to approximate higher order derivatives of y also.

Choose a set of n delays satisfying

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_n.$$

We may approximate the output $y(t)$ around any time $t = t_0$ with the polynomial

$$y_p(t) = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}$$

which interpolates y(t) at the n past instants $t_0-\epsilon\tau_1,\ldots,t_0-\epsilon\tau_n$, i.e.,

$$y_p(t_0-\epsilon\tau_i)=y(t_0-\epsilon\tau_i),\qquad i=\overline{1,n}. \tag{11}$$

Here, $\epsilon>0$ is a small scaling parameter. Since the Vandermonde matrix

$$T(\tau)\triangleq\begin{bmatrix}1 & \tau_1 & \tau_1^2 & \cdots & \tau_1^{n-1}\\ \vdots & & & & \vdots\\ 1 & \tau_n & \tau_n^2 & \cdots & \tau_n^{n-1}\end{bmatrix} \tag{12}$$

is invertible when the delays τ_i are different, (11) can be written in matrix form as

$$\begin{bmatrix}c_0\\ c_1\\ \vdots\\ c_{n-1}\end{bmatrix}=\begin{bmatrix}1 & & & \\ & \frac{1}{(-\epsilon)} & & \\ & & \ddots & \\ & & & \frac{1}{(-\epsilon)^{n-1}}\end{bmatrix}T(\tau)^{-1}\begin{bmatrix}y(t_0-\epsilon\tau_1)\\ y(t_0-\epsilon\tau_2)\\ \vdots\\ y(t_0-\epsilon\tau_n)\end{bmatrix} \tag{13}$$

and we may approximate

$$y^{(i)}(t_0)\approx y_p^{(i)}(t_0)=i!\,c_i\qquad \forall i=\overline{1,n}. \tag{14}$$

This way, the control law (9) at $t=t_0$ can be approximated with

$$u(t_0)=-q_0\,y_p(t_0)-q_1\,y_p'(t_0)-\cdots-q_{n-1}\,y_p^{(n-1)}(t_0).$$

Substituting (13) and (14) into this expression, and applying the same principle for all $t_0>0$ leads to the control law

$$u(t)=-\left[q_0\,\frac{1}{(-\epsilon)}q_1\,\frac{2!}{(-\epsilon)^2}q_2\,\cdots\,\frac{(n-1)!}{(-\epsilon)^{n-1}}q_{n-1}\right]\cdot T(\tau)^{-1}\begin{bmatrix}y(t-\epsilon\tau_1)\\ \vdots\\ y(t-\epsilon\tau_n)\end{bmatrix}. \tag{15}$$

When $\epsilon\to 0+$ the approximation of (9) becomes better and we have the following result.

Proposition 2: Assume that the polynomial $q(\lambda)\triangleq\lambda^n+q_{n-1}\lambda^{n-1}+\ldots+q_0$ is Hurwitz. Assume further that $0\le\tau_1<\tau_2<\cdots<\tau_n$ and let $T(\tau)$ be defined by (12). Then, the control law (15) achieves asymptotic stability for small values of ϵ. Moreover, as $\epsilon\to 0+$, the n rightmost eigenvalues of the closed-loop system converge to the n zeros of $q(\lambda)$.

Proof: With the control law (9), the characteristic equation of the closed-loop system is given by $q(\lambda)=0$, while the control law (15) yields $q_\epsilon(\lambda)=0$, where

$$q_\epsilon(\lambda)=\lambda^n+\left[q_0\,\frac{1}{(-\epsilon)}q_1\,\frac{2!}{(-\epsilon)^2}q_2\,\ldots\,\frac{(n-1)!}{(-\epsilon)^{n-1}}q_{n-1}\right]\cdot T(\tau)^{-1}\begin{bmatrix}e^{-\epsilon\tau_1\lambda}\\ \vdots\\ e^{-\epsilon\tau_n\lambda}\end{bmatrix}. \tag{16}$$

We first establish a relation between $q(\lambda)$ and $q_\epsilon(\lambda)$ as $\epsilon\to 0+$. Therefore, consider an arbitrary $\lambda\in\mathbb{C}$. Using a Taylor expansion. we have

$$e^{-\epsilon\tau_i\lambda}=1+\frac{(-\epsilon\tau_i\lambda)}{1!}+\ldots+\frac{(-\epsilon\tau_i\lambda)^{n-1}}{(n-1)!}+O\left((\epsilon\lambda)^n\right),\qquad i=\overline{1,n}$$

which can be written as

$$\begin{bmatrix}e^{-\epsilon\tau_1\lambda}\\ e^{-\epsilon\tau_2\lambda}\\ \vdots\\ e^{-\epsilon\tau_n\lambda}\end{bmatrix}=T(\tau)\begin{bmatrix}1\\ \frac{(-\epsilon\lambda)}{1!}\\ \vdots\\ \frac{(-\epsilon\lambda)^{n-1}}{(n-1)!}\end{bmatrix}+\begin{bmatrix}O\left((\epsilon\lambda)^n\right)\\ O\left((\epsilon\lambda)^n\right)\\ \vdots\\ O\left((\epsilon\lambda)\right)^n\end{bmatrix}. \tag{17}$$

Substituting (17) into (16) leads to

$$q_\epsilon(\lambda)=q(\lambda)+O(\epsilon\lambda^n). \tag{18}$$

Define a *compact* subset S of the complex plane, which contains all the zeros of $q(s)$. From the expression (18) it follows that the analytic function $q_\epsilon(\lambda)$ *uniformly* converges to $q(\lambda)$ on S as $\epsilon\to 0+$. Therefore, both functions have the same number of zeros in S when ϵ is sufficiently small and moreover, as $\epsilon\to 0+$, the n zeros of $q_\epsilon(\lambda)$ in S converge to n corresponding zeros of $q(\lambda)$. These statements follow from a slight modification of [11, Lemma A1].

The proof is complete when we also show that in any right half plane $q_\epsilon(\lambda)$ has at most n zeros, when ϵ is sufficiently small. This follows from the scaling Property 1: The condition $q_\epsilon(\lambda)=0$ is equivalent with

$$\bar\lambda^n+\left[q_0\epsilon^n\,\frac{\epsilon^{n-1}}{(1)}q_1\,\frac{2!\epsilon^{n-2}}{(-1)^2}q_2\,\cdots\,\frac{(n-1)!\epsilon}{(-1)^{n-1}}q_{n-1}\right]\cdot T(\tau)^{-1}\begin{bmatrix}e^{-\tau_1\bar\lambda}\\ \vdots\\ e^{-\tau_n\bar\lambda}\end{bmatrix}=0$$

where $\bar\lambda=\epsilon\lambda$. This equation can be interpreted as the characteristic equation of a feedback controlled multiple integrator with fixed feedback delays, where the gain can be made arbitrarily small. As proven in [10], n eigenvalues converges to zero as the gain tends to zero, while the real parts of the other eigenvalues move off to minus infinity. This implies that for any $r\in\mathbb{R}$, $q_\epsilon(\lambda)$ has as most n zeros in the half plane $\Re(\bar\lambda)\ge r$ or, equivalently, in $\Re(\lambda)\ge r/\epsilon$, provided ϵ is sufficiently small. □

Remark 1: When the complex variable λ in the characteristic equation of the closed-loop system with control law (9) is formally replaced with $(1-e^{-\lambda\epsilon})/\epsilon$ (except for the term λ^n) and the resulting expression is developed in powers of $e^{-\lambda\epsilon}$, the characteristic equation of a system with $P+$ delay compensator with $(n-1)$ *commensurate* delays is obtained. This is exactly the controller of Proposition 2, when taking one delay equal to zero and the other delays commensurate, i.e. $\tau_i=(i-1)$, $i=\overline{1,n}$.

Note also that a different proof can be carried out using some properties of subharmonics functions [2]. Indeed, by the continuity principle [5], there exists a sufficiently small delay $\epsilon>0$, such that the stability of the "transformed" system obtained using the 'formal modification' suggested before ($\lambda\mapsto(1-e^{-\lambda\epsilon})/\epsilon$) is still preserved, if the 'original' (before transformation) system is stable. Thus, the stability of the closed-loop system using multiple delay blocks follows straightforwardly after $(n-2)$ iterations of the procedure briefly presented.

Remark 2: (Stabilizability with delayed output feedback) In the ODE case, all the output derivatives up to order $(n-1)$ are needed to stabilize a chain of n integrators. However, for a numerical approximation of the $(n-1)$th order derivative at least n sample instants are needed. This indicates that the information obtained from less than n output measurements is *not sufficient* for the necessary complete state reconstruction and, thus, motivates the statement of Conjecture 1.

Using Property 1 the statements of Proposition 2 can be rephrased as follows.

Theorem 1: Assume that $0 \leq \tau_1 < \ldots < \tau_n$ and $q(\lambda)$ Hurwitz. Then, the control law

$$u(t) = -\left[\epsilon^n q_0 \ \frac{\epsilon^{n-1}}{(-1)} q_1 \ \frac{2!\, \epsilon^{n-2}}{(-1)^2} q_2 \ \cdots \ \frac{(n-1)!\, \epsilon}{(-1)^{n-1}} q_{n-1}\right] \cdot T(\tau)^{-1} \begin{bmatrix} y(t-\tau_1) \\ \vdots \\ y(t-\tau_n) \end{bmatrix} \quad (19)$$

achieves asymptotic stability for small values of ϵ. As $\epsilon \to 0+$, the n rightmost eigenvalues converge to $\epsilon\lambda_i$, $i = \overline{1,n}$, with λ_i the zeros of $q(\lambda)$.

In the next paragraph, we outline an alternative approach to design a stabilizing feedback law.

B. Control Laws Based on Exact Pole Placement and Low-Gain Design

In the control law

$$u(t) = -\sum_{j=1}^{n} k_j y(t-\tau_j)$$

there are n degrees of freedom, which allow to place n closed-loop eigenvalues at prescribed values. This way stability cannot be insured in general because the number of eigenvalues is infinite and only n of them are controlled. However, this conflict can be solved when using the low-gain approach, developed in [9], [10] in the context of the stabilization of integrators with an input delay and input constraints. The basic idea is as follows. When the controlled eigenvalues are placed close to zero, we expect the gains to be low. However, when the gains tend to zero, all the eigenvalues, except of n, are shifted far away in the left half plane, because the DDE behaves as an ODE with a vanishing (delayed) perturbation. We now illustrate this approach with an example, where n closed-loop eigenvalues are placed at the same position, because this gives rise to an explicit formula with a structure analogous to (19):

Theorem 2: Assume $0 \leq \tau_1 < \tau_2 \cdots < \tau_n$ and let $T(\tau)$ be defined by (12). Then, control law

$$u(t) = (-1)^n [\epsilon^n \ n\epsilon^{n-1} \ldots n!\, \epsilon] \cdot T(\tau)^{-1} \begin{bmatrix} e^{-\epsilon\tau_1} & & \\ & \ddots & \\ & & e^{-\epsilon\tau_n} \end{bmatrix} \begin{bmatrix} y(t-\tau_1) \\ \vdots \\ y(t-\tau_n) \end{bmatrix} \quad (20)$$

achieves asymptotic stability for small values of ϵ. Moreover, there is a closed-loop eigenvalue at $\lambda = -\epsilon$ with multiplicity n.

Proof: The characteristic equation of the closed-loop system is given by

$$p(\lambda) \triangleq \lambda^n + \sum_{j=1}^{n} k_j e^{-\lambda\tau_j} = 0.$$

Assigning n eigenvalues to $\lambda = \bar{\lambda}$ yields the conditions $p(\bar{\lambda}) = 0, \ldots, p^{n-1}(\bar{\lambda}) = 0$, or

$$\begin{bmatrix} e^{-\bar{\lambda}\tau_1} & \cdots & e^{-\bar{\lambda}\tau_n} \\ -\tau_1 e^{-\bar{\lambda}\tau_1} & \cdots & -\tau_n e^{-\bar{\lambda}\tau_n} \\ \vdots & & \vdots \\ (-\tau_1)^{n-1} e^{-\bar{\lambda}\tau_1} & \cdots & (-\tau_n)^{n-1} e^{-\bar{\lambda}\tau_n} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = -\begin{bmatrix} \bar{\lambda}^n \\ n\bar{\lambda}^{n-1} \\ \vdots \\ n(n-1)\cdots 2\bar{\lambda} \end{bmatrix}.$$

This can be written as

$$\begin{bmatrix} 1 & & & \\ & (-1) & & \\ & & \ddots & \\ & & & (-1)^{n-1} \end{bmatrix} T(\tau)^T \cdot \begin{bmatrix} e^{-\bar{\lambda}\tau_1} & & \\ & \ddots & \\ & & e^{-\bar{\lambda}\tau_n} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = -\begin{bmatrix} \bar{\lambda}^n \\ n\bar{\lambda}^{n-1} \\ \vdots \\ n(n-1)\cdots 2\bar{\lambda} \end{bmatrix}$$

and, therefore

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = -\begin{bmatrix} e^{\bar{\lambda}\tau_1} & & \\ & \ddots & \\ & & e^{\bar{\lambda}\tau_n} \end{bmatrix} T(\tau)^{-T} \begin{bmatrix} \bar{\lambda}^n \\ (-1)n\bar{\lambda}^{n-1} \\ \vdots \\ (-1)^{n-1} n(n-1)\cdots 2\bar{\lambda} \end{bmatrix}.$$

Choosing $\bar{\lambda} = -\epsilon$ leads to the control law (20). When we let $\epsilon \to 0+$, we have

$$K(\epsilon) = [k_1(\epsilon) \ \cdots \ k_n(\epsilon)]^T \to 0.$$

Hence, the n eigenvalues at zero of the uncontrolled system are shifted to $-\epsilon$, while other eigenvalues cannot cause instability when ϵ (i.e., $K(\epsilon)$) is sufficiently small. □

Remark 3: For $q(\lambda) = (\lambda+1)^n$ the control law (19) reduces to $u(t) =$

$$u(t) = -\left[\epsilon^n \ \frac{n\epsilon^{n-1}}{(-1)} \ \frac{n(n-1)\epsilon^{n-2}}{(-1)^2} \ \cdots \ \frac{n!\, \epsilon}{(-1)^{n-1}}\right] \cdot T(\tau)^{-1} \begin{bmatrix} y(t-\tau_1) \\ \vdots \\ y(t-\tau_n) \end{bmatrix}. \quad (21)$$

This control law does not coincide with (20) because it is based on an asymptotic *approximation* of $q(\lambda)$, while (20) is based on an *exact* placement of n eigenvalues.

Remark 4: Both Theorems 1 and 2 guarantee asymptotic stability for sufficiently small values of ϵ. A threshold can be computed by performing a numerical continuation of the closed-loop eigenvalues as a function of the parameter ϵ, as illustrated in the next section. Even when the structure of (19) or (20) is not explicitly used, a stabilizing feedback law may still be synthesized by means of the so-called *continuous pole placement method* [12].

IV. Example

For the *triple integrator*, the control law (19) with $q(\lambda) = (\lambda+1)^3$ and $\tau_i = (i-1)$, $\overline{i = 1,3}$ takes the form

$$u(t) = \left(-3\epsilon - \frac{9}{2}\epsilon^2 - \epsilon^3\right) y(t) + (6\epsilon + 6\epsilon^2) y(t-1) + \left(-3\epsilon - \frac{3}{2}\epsilon^2\right) y(t-2). \quad (22)$$

In Fig. 1(top) the rightmost eigenvalues of the closed-loop system are displayed as a function of the parameter ϵ. For $\epsilon < \bar{\epsilon}$, indicated on the figure, the closed-loop system is asymptotically stable. According to Theorem 1, the three dominant eigenvalues converge to $\lambda = -\epsilon$ as $\epsilon \to 0+$.

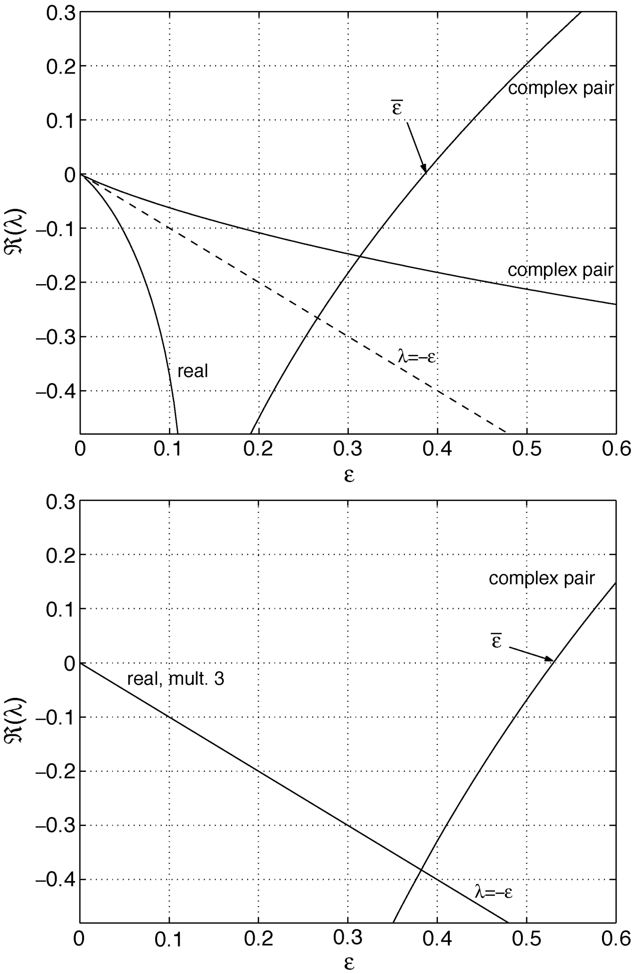

Fig. 1. Real parts of the closed-loop eigenvalues of the triple integrator, controlled with (22) (top), respectively (23) (bottom). The calculations were done with the software package DDE-BIFTOOL [6].

The control law (20) with $\tau_i = (i-1)$, $\overline{i=1,3}$ is given by

$$u(t) = \left(-3\epsilon + \frac{9}{2}\epsilon^2 - \epsilon^3\right) y(t) + (6\epsilon - 6\epsilon^2)e^{-\epsilon}y(t-1) + \left(-3\epsilon + \frac{3}{2}\epsilon^2\right) e^{-2\epsilon}y(t-2) \quad (23)$$

and the closed-loop eigenvalues are shown in Fig. 1(bottom). Following Theorem 2, three eigenvalues lie at $\lambda = -\epsilon$, for all values of ϵ. Note that when ϵ is small, the spectrum is similar to the previous case, which is not surprising since the dominant terms in (22) and (23) (the terms $\sim \epsilon$) are equal. Simulations of trajectories show that the performances of controllers (22) and (23) are comparable, except when ϵ is close to the stability limit $\bar{\epsilon}$ of (22). This is not surprising, because both controllers have the property that, for small ϵ, the closed-loop dynamics are determined by three dominant eigenvalues, while the other eigenvalues are far away in the left-half plane. However, an important difference is that, in contrast to (23), the control law (22) only involves an *approximate* assignment of these three dominant eigenvalues at $\lambda = -\epsilon$, where the relative error converges slowly[2] to zero as $\epsilon \to 0+$. This can be seen in Fig. 1(top). In general, for the nth-order integrator, it may be desirable from a performance point of view to assign the n rightmost eigenvalues to $(\epsilon\lambda_1, \ldots, \epsilon\lambda_n)$, where λ_i, $i = \overline{1,n}$ are the zeros of an *a priori* chosen polynomial $q(\lambda)$ and ϵ a scaling parameter. For a given set of delays, formula (19) is readily applicable in such a case. A drawback, however, is that ϵ should not only be tuned to have only n dominant eigenvalues, but also to insure that the latter are sufficiently close to their desired values $\epsilon\lambda_i$, which may lead to unnecessarily small gains. This problem does not occur when using the approach proposed in Section III-B, which is based on an exact placement of the rightmost eigenvalues. Here, the drawback is that, although the gains can be computed numerically for any polynomial $q(\lambda)$ and any value of ϵ, an explicit formula of a by ϵ parametrized family of control laws only exists for special cases of $q(\lambda)$ like $q(\lambda) = (\lambda+1)^n$, which led to expression (20).

Finally, recall that when ϵ is fixed to a value smaller than $\bar{\epsilon}$, (22) and (23) actually define a whole one-parameter family of stabilizing feedback laws, by making use of the Property 1.

[2]This is especially the case for $q(\lambda) = (\lambda+1)^3$, because eigenvalues with a multiplicity larger than one are generally very sensitive to changes of the system's parameters.

V. Concluding Remarks

We addressed the stabilization problem of a chain of integrators using multiple delays in the control law. More precisely, we have proved that a chain of n integrators can be stabilized by a proportional+delay controller including $(n-1)$ delays, or by a chain of n delay blocks. The proposed proofs are constructive, and purpose two different ways to handle the same problem. An explicit construction of the corresponding controllers was proposed for the chain including *three integrators*.

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A VSC Approach for the Robust Stabilization of Nonlinear Plants With Uncertain Nonsmooth Actuator Nonlinearities—A Unified Framework

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***Abstract*—This note addresses the stabilization problem of an uncertain intrinsically nonlinear single-input–single-output plant containing nonsmooth nonlinearities (dead zone, backlash, hysteresis) in the actuator device. A unified framework for its solution is here proposed, assuming that the parameters of the nonlinearities are uncertain as well. To this purpose, the hysteresis model used in a previous paper has been modified into an "extended" one, and a robust control law ensuring asymptotic stabilization has been synthesized using it. The resulting controller has been shown to be a full generalization of previous results (it includes, as particular cases, control laws previously developed for backlash and dead zone), ensuring also that the inner "forbidden" part of nonlinearity characteristics is never entered, even in the presence of uncertainties. Theoretical results have been validated by simulation on a simple mechanical system.**

***Index Terms*—Hysteresis, nonlinear systems, nonsmooth actuator nonlinearity, robust control, sliding-mode control.**

I. INTRODUCTION

In real control systems, actuators, sensors and, more in general, a wide range of physical devices contain "nonsmooth" nonlinearities, such as backlash, dead zone or hysteresis. Due to physical imperfections, indeed, such nonlinearities are always present in real plants, particularly in mechanical systems (e.g., the drive train in cars, rolling mills, printing presses, industrial robots [3]). Nevertheless, control design techniques usually applied in practice do often ignore the presence of such nonlinearities in system components. Indeed, much of the recent advances in robust and nonlinear control would be otherwise not applicable, due to the nondifferentiable and nonmemoryless character of those nonlinearities. As a consequence, system performance may result severely deteriorated, showing oscillations, delays and inaccuracy (for example, servomechanisms usually require complete elimination of backlash to work properly). It can be claimed that "actuator and sensor nonlinearities are among the key factors limiting both static and dynamic performance of feedback control systems" [4].

Hysteresis phenomena usually show up in electromagnetic fields, electronic relay circuits, mechanical actuators [5]. As well known, the area enclosed by the loop is often thought of as representing energy loss, hence it is important to be able to avoid such a condition. As a matter of fact, finding a general model of hysteresis is still a debated research issue, due to the phenomena complexity. A number of different models are available in the literature (for a complete survey, see [6] and [7]). In most cases, however, rigorous mathematical models of hysteresis tend to be very complicated and are hardly suited for controller design. A well assessed hysteresis model, capturing most of its characteristics, still retaining some simplicity being piecewise linear, has been proposed in [1] and [4]. Even in simple cases, however, traditional control methods fail and new approaches claim for being developed [8], relevant uncertainty being also necessary to be accounted for in the nonlinearity models parameters. It follows that the development of control techniques coping with unknown hystereses is a challenging issue, which recently re-attracted significant attention due to their possible applications in the so called smart actuators.

To address such a challenge, a number of different approaches have been proposed, in most cases restricted to backlash. Just to name a few, neural networks [9], dithering [10], fuzzy logic [11] and optimal control [12] have been used to compensate for nonlinearities present in the actuator. An important research thrust, dealing with unknown nonlinearities cascaded to linear plants, is based on adaptive control [4]. The main idea underlying this approach is the introduction of an inverse model of the actuator nonlinearity, updated adaptively, aimed at cancelling its effects. It is worth noting, however, that when the adaptive inverse is used for control, the effect of hysteresis may not be completely cancelled [1].

Variable structure control (VSC) techniques have been used as well [2], [13], [14]. Indeed, the well-known robustness features and the discontinuous character of sliding mode control [15] appears particularly well suited for handling intrinsically nonlinear and uncertain single-input–single-output (SISO) plants containing nonsmooth nonlinearities described using piece-wise linear functions. Moreover, model inversion is not required, to avoid the possible amplification of additive measurement disturbances which may result from inversion of the output nonlinearity.

Following the lines of [2] and [16], a sliding-mode approach is presented in this note in order to solve the stabilization problem for an intrinsically nonlinear SISO plant with an uncertain hysteresis-like nonlinearity in the actuator. With the aim of attaining a controller general enough to include, as particular cases, the laws [2] previously developed for backlash and dead zone, the hysteresis model used in [1] has been modified into an "extended" one. Using it, a robust control law ensuring asymptotic stabilization has been synthesized. The developed controller has been shown to be a full generalization of previous results, ensuring also that the inner "forbidden" part of nonlinearity characteristics is never entered, even in the presence of uncertainties.

II. SYSTEM MODEL AND PROBLEM STATEMENT

Consider an uncertain single-input intrinsically nonlinear system

$$\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}) + \Delta\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u + \mathbf{d}(\mathbf{x}, t) \quad (1)$$

$$u = f(v) \quad (2)$$

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