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Automatica 41 (2005) 991–998

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Brief paper

An eigenvalue based approach for the stabilization of linear time-delay systems of neutral type☆

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Received 15 March 2004; received in revised form 15 November 2004; accepted 22 November 2004
Available online 25 February 2005

Abstract

An eigenvalue based approach for the stabilization of linear neutral functional differential equations is presented, which extends the recently developed continuous pole placement method for delay equations of retarded type. The approach consists of two steps. First the stability of the associated difference equation is determined and a procedure is applied to compute the supremum of the real parts of its characteristic roots, which corresponds to computing the radius of the essential spectrum of the solution operator of the neutral equation. No restrictions are made on the dimension of the system and the number of delays. Also the effect of small delay perturbations is explicitly taken into account. As a result of this first step the stabilization problem of the neutral equation is reduced to a problem involving only a finite number of characteristic roots. As a second step, stabilization is achieved by shifting the rightmost or unstable characteristic roots to the left half plane in a quasi-continuous way, by applying small changes to the controller parameters, and meanwhile monitoring other characteristic roots with a large real part. A numerical example is presented.

Keywords: Time-delay systems; Neutral type systems; Stability

1. Introduction

Many engineering systems can be modeled by delay differential equations of neutral type, for instance lossless transmission lines (Kolmanovskii & Nosov, 1986) and partial element equivalent circuits (PEECs) (Bellen, Guglielmi, & Ruehli, 1999) in electrical engineering, combustion systems (Murray et al., 1998) and controlled constrained manipulators in mechanical engineering (Niculescu & Brogliato, 1999). They also arise in certain implementation schemes of predictive controllers, see e.g. Engelborghs, Dambrine, and Roose (2001).

This note deals with the asymptotic stabilization of the neutral system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)-\sum_{k=1}^{m} H_k x(t-\tau_k)\right) = A_0\, x(t)+\sum_{k=1}^{m} A_k x(t-\tau_k)+B_0 u(t) + \sum_{k=1}^{m} B_k u(t-\tau_k), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ are the inputs and $0<\tau_1<\tau_2\cdots<\tau_m$ represent the delays in both state and inputs. We use a state feedback controller of the form

$$u(t) = Kx(t). \tag{2}$$

Notice that class (1)–(2) is quite general. It allows us for instance also to treat systems where the control law is a

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Didier Henrion under the direction of Editor R. Tempo.

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0005-1098/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2004.11.032

linear combination of the present state and/or delayed states. For example, the control system

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - cx(t-\tau_1)) = ax(t) + bu(t),$$
$$u(t) = k_1 x(t-\tau_2) + k_2 x(t-\tau_3), \quad x \in \mathbb{R}, \ u \in \mathbb{R} \tag{3}$$

has the same closed-loop description as the following two-input control system of the form (1) and (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - cx(t-\tau_1))$$
$$= ax(t) + [b\ 0]v(t-\tau_2) + [0\ b]v(t-\tau_3),$$
$$v(t) = [k_1\ \ k_2]^T x(t), \quad x \in \mathbb{R}, \ v \in \mathbb{R}^2. \tag{4}$$

Also certain classes of systems with distributed delays can be brought into the standard form (1)–(2) by a model transformation, see e.g. Luzyanina and Roose (2004) for the theory and Michiels, Engelborghs, Vansevenant, and Roose (2002b) for an application to stabilization.

The controller design methodology extends the eigenvalue based *continuous pole placement method*, developed in Michiels et al. (2002b) for the stabilization of linear time-delay systems of *retarded type*. The problems under consideration in Michiels et al. (2002b) are all characterized by an *infinite numbers* of characteristic roots, whereas the number of controller parameters is *finite*. The methodology consists of controlling the rightmost or unstable characteristic roots only, which are shifted to the left half plane in a quasi-continuous way, by applying small changes to the controller parameters, and meanwhile *monitoring* the other characteristic roots with a larger real part. Its main ingredients are therefore (i) computing numerically the rightmost, stability determining characteristic roots and (ii) a procedure to shift these roots. The difference with the related approach of Zítek and Vyhlídal (2000, 2002) is the adaptation of the controller parameters in small steps making it possible to monitor the behavior of uncontrolled characteristic roots and *react* on it, whereas in Zítek and Vyhlídal (2000, 2002) one directly computes controller parameters, which assign a finite number of characteristic roots to prescribed positions. That way the position of the other characteristic roots is only checked a posteriori, which could lead to some trial and error in the design procedure.

As alternative eigenvalue based controller design methods for delay systems, we mention algebraic approaches, see e.g. Emre and Khargonekar (1982); Lu, Lee, and Zak (1986); Morse (1976); Sename, Lafay, and Rabah (1995); Watanabe (1995) and the references therein. For instance, one can consider a delay system as a system over a ring (Morse, 1976), where the system matrices depend on an indeterminate corresponding to the delay (shift) operator. Starting from structural properties (as controllability notions) one typically derives stabilizability conditions and associated controller structures. For instance, for systems of retarded type with input delay, spectral controllability implies the existence of controllers which eliminate the delay from the closed-loop characteristic equation and results in a finite number of freely assignable characteristic roots (Watanabe, 1995). In general, algebraic methods are powerful but the stabilizability conditions obtained may be very stringent (leading to restrictions on the system under consideration) and the resulting controllers may be complex and hard to implement. The latter is also the case for controllers based on spectral decomposition (Hale & Verduyn Lunel, 1993), which can shift a finite number of unstable roots while keeping the others invariant, see Pandolfi (1976), because such control laws contain distributed delay terms. The continuous pole placement procedure, on the other hand, starts from any a priori determined controller structure, as (2), where some controller parameters need to be tuned. Of course, the existence of and thus the convergence to a stabilizing solution heavily depends on the chosen controller structure. However, as a major advantage, the methodology is *generally applicable* (systems with multiple dependent/independent delays in input, output, state, control, etc.), because the applicability only relies on the continuous dependence of characteristic roots w.r.t. controller parameters and the computation of characteristic roots (see Michiels et al., 2002b for motivating applications). Moreover, when a simple controller structure is chosen, as (2) or a classical PID, and if stabilizing controller parameters are found, the resulting controller may have an advantage from an implementation and robustness point of view, in spite of the fact that e.g. not the whole spectrum is freely assignable (see Michiels et al., 2002b for a comparison between a finite spectrum assignment controller and instantaneous state feedback for an input delay systems). As we shall see, also the robustness of stability w.r.t. delay perturbations can be taken into account explicitly, being an important issue for neutral equations.

The neutral type of system (1)–(2) induces complications compared to the retarded case, which makes adaptation of the continuous pole placement algorithm necessary. Whereas for systems of retarded type the number of characteristic roots in any right half plane is always finite, giving the control problem a finite dimensional nature, systems of neutral type may exhibit chains of characteristic roots, whose imaginary parts tend to infinity, yet whose real parts have a finite limit (Hale & Verduyn Lunel, 1993; Kolmanovskii & Nosov, 1986). In addition, such high frequency roots may be very sensitive to parameter changes, in particular to delay changes, and may even cause a sensitivity of stability to infinitesimal delay changes (Michiels, Engelborghs, Roose, & Dochain, 2002a). The presence of highly sensitive high frequency modes makes the determination of stability properties by numerically computing a finite number of characteristic roots or by computing all characteristic roots in a compact set unsafe. Therefore, a *two-step* approach will be proposed. First a numerical procedure is developed to compute the supremum of the real parts of the characteristic roots of the associated difference equation, which corresponds to computing the radius of the essential spectrum of the solution operator of the neutral equation. As a consequence, only a finite number of characteristic roots in a compact set need to be computed to obtain precise stability information and, furthermore, the shifting procedure of the continuous pole placement method of Michiels et al. (2002b) to compute a stabilizing feedback gain K becomes applicable. This is done as a second step.

The structure of the paper is as follows: after some preliminaries we comment on relevant spectral properties of neutral equations and derive a procedure to compute the radius of the essential spectrum of the solution operator. Then we outline the adaptation of the continuous pole placement algorithm of Michiels et al. (2002b) to the neutral case. The paper ends with an illustrative example.

2. Preliminaries

We recall some basic notions on stability of neutral equations, based on Hale and Verduyn Lunel (1993, 2002b, 2003), and introduce notations.

Let $\vec{\tau} = (\tau_1, \ldots, \tau_m)$. The initial condition for the neutral system (1)–(2) is a function segment $\phi \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n)$, where $\mathscr{C}([-\tau_m, 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-\tau_m, 0]$ into $\mathbb{R}^n$ and equipped with the supremum-norm. The fact that the map $\mathscr{N}: \mathscr{C}([-\tau_m, 0], \mathbb{R}^n) \to \mathbb{R}^n$, defined by

$$\mathscr{N}(\phi) = \phi(0) - \sum_{k=1}^{m} H_k \phi(-\tau_k)$$

is atomic at zero guarantees existence and uniqueness of solutions of (1)–(2). Let $x(\phi) : t \in [-\tau_m, \infty) \to x(\phi)(t) \in \mathbb{R}^n$ be the unique forward solution with initial condition $\phi \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n)$, i.e. $x(\phi)(\theta) = \phi(\theta),\ \forall \theta \in [-\tau_m, 0]$. Then the state at time t is given by the function segment $x_t(\phi) \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n)$ defined as $x_t(\phi)(\theta) = x(\phi)(t + \theta),\ \theta \in [-\tau_m, 0]$. Denote with $T(t)$ the solution operator, mapping initial data onto the state at time t, i.e.

$$(T(t)\phi)(\theta) = x_t(\phi)(\theta) = x(\phi)(t + \theta), \quad \theta \in [-\tau_m, 0]. \tag{5}$$

This is a strongly continuous semi-group. The associated difference equation of (1)–(2) is given by $\mathscr{N}(x_t) = 0$, i.e.

$$x(t) - \sum_{k=1}^{m} H_k x(t - \tau_k) = 0. \tag{6}$$

For any initial condition $\phi \in \mathscr{C}_{\mathrm{D}}([-\tau, 0], \mathbb{R}^n)$, where

$$\mathscr{C}_{\mathrm{D}}([-\tau_m, 0], \mathbb{R}^n) = \{\phi \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n) : \mathscr{N}(\phi) = 0\}$$

a solution of (6) is uniquely defined. Let $T_{\mathrm{D}}(t)$ be the corresponding solution operator.

The asymptotic behavior of the solutions and, thus, the stability of the null solution of the neutral equation (1)–(2) is determined by the spectral radius $r_\sigma(T(t))$, satisfying

$$r_\sigma(T(1)) = \mathrm{e}^{c_{\mathrm{N}}}, \quad c_{\mathrm{N}} = \sup\{\Re(\lambda) : \det(\Delta_{\mathrm{N}}(\lambda)) = 0\}, \tag{7}$$

where the characteristic matrix Δ_{N} is given by

$$\Delta_{\mathrm{N}}(\lambda) = \left(\lambda \Delta_{\mathrm{D}}(\lambda) - A_0 - B_0 K - \sum_{k=1}^{m} (A_k + B_k K)\mathrm{e}^{-\lambda \tau_k}\right) \tag{8}$$

and

$$\Delta_{\mathrm{D}}(\lambda) = \left(I - \sum_{k=1}^{m} H_k \mathrm{e}^{-\lambda \tau_k}\right).$$

For instance, the null solution is exponentially stable iff $r_\sigma(T(1)) < 1$ or equivalently $c_{\mathrm{N}} < 0$ (see Hale & Verduyn Lunel, 1993 for an overview of stability definitions and their relation to spectral properties). In a similar way, the stability of the difference equation (6) is determined by the spectral radius

$$r_\sigma(T_{\mathrm{D}}(1)) = \mathrm{e}^{c_{\mathrm{D}}}, \quad c_{\mathrm{D}} = \sup\{\Re(\lambda) : \det(\Delta_{\mathrm{D}}(\lambda)) = 0\}. \tag{9}$$

An important property in the stability analysis of neutral equations is the relation

$$r_{\mathrm{e}}(T(1)) = r_\sigma(T_{\mathrm{D}}(1)), \tag{10}$$

where $r_{\mathrm{e}}(.)$ denotes the radius of the essential spectrum, see e.g. Hale and Verduyn Lunel (1993) and Hale (1995). From this follows the well known result that a *necessary condition* for the exponential stability of the null solution of (1)–(2) is given by the exponential stability of the null solution of the difference equation (6).

In the rest of the paper we will call the solutions of $\det(\Delta_{\mathrm{N}}(\lambda)) = 0$ the characteristic roots of the neutral system (1)–(2). Analogously we will call the solutions of $\det(\Delta_{\mathrm{D}}(\lambda)) = 0$ the characteristic roots of the difference equation (6).

3. Spectral properties

3.1. Strong stability of the difference equation

It is well known that the spectral radius (9), although continuous in the system matrices H_k, is *not* continuous in the *delays* $\vec{\tau}$, see e.g. Hale and Verduyn Lunel (1993, 2002a) and Henry (1974), which carries over to

$$c_{\mathrm{D}}(\vec{\tau}) = \sup\left\{\Re(\lambda) : \det\left(I - \sum_{k=1}^{m} H_k \mathrm{e}^{-\lambda \tau_k}\right) = 0\right\}. \tag{11}$$

One consequence of the non-continuity of c_{D} w.r.t. $\vec{\tau}$ is that arbitrarily small perturbations on the delays may destroy stability of the difference equation. This has led to the introduction of the concept of *strong stability* in Hale and Verduyn Lunel (2002a): we say that the null solution of Eq. (6) is strongly exponentially stable if it remains exponentially stable when subjected to small variations in the delays. Theorem 2.2 and Corollary 2.2 of Hale and Verduyn Lunel (2002a) provide the following conditions.

Proposition 1. *The null solution of the delay difference equation* (6) *is strongly exponentially stable if and only if* $\gamma_0 < 1$, *where*

$$\gamma_0 := \max_{\vec{\theta} \in [0,\ 2\pi]^m} r_\sigma \left(\sum_{k=1}^{m} H_k e^{i\theta_k} \right).$$

Furthermore, if $\gamma_0 > 1$ *then Eq.* (6) *is exponentially unstable for rationally independent*[1] *delays.*

Notice that the quantity γ_0 does not depend on the value of the delays, i.e. exponential stability locally in the delays is equivalent with exponential stability globally in the delays (Hale & Verduyn Lunel, 2002a).

3.2. Computing bounds on the real parts of the characteristic roots

When the difference equation is strongly stable, it will still be very useful in general and in particular for the continuous pole placement procedure to have *more precise* information about the position of the real parts of its characteristic roots, and in particular the upper bound (11). Due to *lack of continuity* of this quantity w.r.t. the delays we are from a practical point of view once again led to the smallest upper bound, which is '*insensitive*' to small delay changes. More precisely, we define this 'safe' upper bound $\bar{C}_D(\vec{\tau})$ as follows.

Definition 2. Let $\bar{C}_D(\vec{\tau}) \in \mathbb{R}$ be defined as

$$\bar{C}_D(\vec{\tau}) = \lim_{\varepsilon \to 0+} c_\varepsilon(\vec{\tau}),$$

where

$$c_\varepsilon(\vec{\tau}) = \sup\{c_D(\vec{\tau} + \delta\vec{\tau}) : \delta\vec{\tau} \in \mathbb{R}^m \text{ and } \|\delta\vec{\tau}\| \leqslant \varepsilon\}.$$

Clearly we have $\bar{C}_D(\vec{\tau}) \geqslant c_D(\vec{\tau})$, and as we shall illustrate at the end of the section the inequality can be *strict*. From Definition 2 it follows that

Proposition 3. $\bar{C}_D(\vec{\tau})$ *is the smallest real number satisfying the following property*:

$$\forall \mu > 0, \quad \exists \nu > 0 \quad \text{s.t.} \quad c_D(\vec{\tau} + \delta\vec{\tau}) < \bar{C}_D(\vec{\tau}) + \mu, \\ \forall \delta\vec{\tau} \in \mathbb{R}^m \quad \text{with } \|\delta\vec{\tau}\| < \nu. \tag{12}$$

[1] The m components of $\vec{\tau} = (\tau_1, \dots, \tau_m)$ are rationally independent if and only if $\sum_{k=1}^{m} n_k \tau_k = 0,\ n_k \in \mathbb{Z}$ implies $n_k = 0,\ \forall k = 1, \dots, m$. For instance, two delays τ_1 and τ_2 are rationally independent if their ratio is an irrational number.

In order to find an analytical expression for $\bar{C}_D(\vec{\tau})$, let us define the function $f : \mathbb{R} \to \mathbb{R}_+$ by

$$f(c;\ \vec{\tau}) = \max_{\vec{\theta} \in [0, 2\pi]^m} r_\sigma \left(\sum_{k=1}^{m} H_k e^{-c\tau_k} e^{i\theta_k} \right) \tag{13}$$

which is *continuous* in both its *argument* c and *parameters* $\vec{\tau}$. Notice that $\gamma_0 = f(0;\ \vec{\tau})$. We first prove two technical lemmas. A direct generalization of Proposition 1 yields

Lemma 4. *Let the function f be defined by* (13) *and* $c \in \mathbb{R}$. *If* $f(c;\ \vec{\tau}) < 1$ *then there exists a number* $\varepsilon > 0$ *such that*

$$c_D(\vec{\tau} + \delta\vec{\tau}) < c, \quad \forall \delta\vec{\tau} \in \mathbb{R}^m \quad \text{with } \|\delta\vec{\tau}\| < \varepsilon.$$

Furthermore, if $f(c;\ \vec{\tau}) > 1$ *then there exist arbitrarily small delay perturbations* $\delta\vec{\tau}$ *such that* $c_D(\vec{\tau} + \delta\vec{\tau}) > c$.

Proof. When substituting $\lambda = \bar{\lambda} + c$ in the characteristic equation of (6) one obtains

$$\det \left(I - \sum_{k=1}^{m} (H_k e^{-c\tau_k}) e^{-\bar{\lambda}\tau_k} \right) = 0, \tag{14}$$

the characteristic equation of

$$x(t) - \sum_{k=1}^{m} H_k e^{-c\tau_k} x(t - \tau_k) = 0. \tag{15}$$

If $f(c;\ \vec{\tau}) < 1$, the continuity of this quantity w.r.t. $\vec{\tau}$ implies the existence of a number $\varepsilon > 0$ such that $f(c;\ \tau + \delta\vec{\tau}) < 1,\ \forall \delta\vec{\tau} \in \mathbb{R}^m$ with $\|\delta\vec{\tau}\| < \varepsilon$. Taking this fact into account an application of Proposition 1 to (15) yields that (15) is exponentially stable and remains so under small perturbations of $\vec{\tau}$, which is equivalent to the first assertion of the lemma.

If $f(c;\ \vec{\tau}) > 1$, then for all $\varepsilon > 0$ there exists a $\delta\vec{\tau} \in \mathbb{R}^m$ such that $\|\delta\vec{\tau}\| < \varepsilon$, $f(c;\ \vec{\tau} + \delta\vec{\tau}) > 1$ (continuity argument) and such that the components of $\vec{\tau} + \delta\vec{\tau}$ are rationally independent (the set of rationally independent numbers is dense in $\mathbb{R}^m$). When perturbing the delays in (15) to $\vec{\tau} + \delta\vec{\tau}$, (15) becomes exponentially unstable, following from an application of the second statement of Proposition 1. This corresponds the second assertion. □

Lemma 5. *The function f is strictly decreasing.*

Proof. The proof is by contradiction. First assume that for some real numbers $c_1 < c_2$, we would have

$$f(c_1;\ \vec{\tau}) < f(c_2;\ \vec{\tau}). \tag{16}$$

Let $\alpha \in \mathbb{R}$ satisfy $f(c_1;\ \vec{\tau}) < \alpha < f(c_2;\ \vec{\tau})$. Then $f(c_1;\ \vec{\tau})/\alpha < 1$ and $f(c_2;\ \vec{\tau})/\alpha > 1$, i.e.

$$\max_{\vec{\theta} \in [0,\ 2\pi]^m} r_\sigma \left(\sum_{k=1}^{m} \frac{H_k}{\alpha} e^{-c\tau_k} e^{i\theta_k} \right) \begin{cases} < 1, & c = c_1, \\ > 1, & c = c_2. \end{cases} \tag{17}$$

When applying Lemma 4 to the difference equation

$$x(t)-\sum_{k=1}^{m}\frac{H_k}{\alpha}x(t-\tau_k)=0 \tag{18}$$

the two statements (17) lead to a contradiction.

Next we treat the case where $f(c_1;\ \vec{\tau})=f(c_2;\ \vec{\tau}) :=\alpha$ for $c_1<c_2$. Based on the above scaling argument we can assume without loosing generality that $\alpha=1$. Because $f(c_2;\ \vec{\tau})=1$ there exist numbers $(\theta_1,\ldots,\theta_m)\in[0,\ 2\pi]^m$ such that

$$\det\left(I-\sum_{k=1}^{m}H_k\mathrm{e}^{-c_2\tau_k}\mathrm{e}^{\mathrm{i}\theta_k}\right)=0.$$

The following property follows (see e.g. Michiels et al., 2002a, Theorem 2.2):

$$\forall\varepsilon>0,\quad \exists\delta\vec{\tau}\in\mathbb{R}^m,\quad \text{with } \|\delta\vec{\tau}\|<\varepsilon \quad\text{and}\quad \exists\lambda\in\mathbb{C}\text{ s.t.}$$
$$|\Re(\lambda)-c_2|<\varepsilon,\quad \det\left(I-\sum_{k=1}^{m}H_k\mathrm{e}^{-\lambda(\tau_k+\delta\tau_k)}\right)=0,$$

i.e. by perturbing the delays one can always find characteristic roots with real part arbitrarily close to c_2. By using the approximation and continuation arguments spelled out in the proof of Hale and Verduyn Lunel (2002a, Lemma 2.2) this fact implies that $f(c_1;\ \vec{\tau})>1$ and we have once again a contradiction. □

Combining the above results we arrive at the main result of this section, which lays the basis for the numerical computation of $\bar{C}_{\mathrm{D}}(\vec{\tau})$:

Theorem 6. *The quantity $\bar{C}_{\mathrm{D}}(\vec{\tau})$ of Definition* 2 *is the unique zero of the strictly decreasing function*

$$c\in\mathbb{R}\rightarrow f(c;\ \vec{\tau})-1,$$

where f is defined in (13). *Furthermore, $\bar{C}_{\mathrm{D}}(\vec{\tau})$ is continuous in the delays $\vec{\tau}\in\mathbb{R}_+^m$.*

Proof. The presence[2] of a characteristic root of (6), say $\lambda=\tilde{c}+\tilde{d}i$, implies that

$$r_\sigma\left(\sum_{k=1}^{m}H_k\mathrm{e}^{-\tilde{c}\tau_k}\mathrm{e}^{-\mathrm{i}\tilde{d}\tau_k}\right)=1$$

and thus $f(\tilde{c};\ \vec{\tau})\geqslant 1$. Since f is continuous and $\lim_{c\rightarrow\infty}f(c;\ \vec{\tau})=0$, one can conclude from Lemma 5 that there is a unique zero of $f(c;\ \vec{\tau})-1$.

If $f(c;\ \vec{\tau})>1$ then by continuity there exists a number $c_{\mathrm{u}}>c$ such that $f(c_{\mathrm{u}};\ \vec{\tau})>1$. By the second assertion of Lemma 4, there are infinitesimal delay perturbations $\delta\vec{\tau}$ such that

$$c_{\mathrm{D}}(\vec{\tau}+\delta\vec{\tau})>c_{\mathrm{u}}>c.$$

[2] When there are no characteristic roots, we are in the degenerate case where $\Delta_{\mathrm{D}}(\lambda)$ is *unimodular*. From (8) the closed-loop system (1)–(2) is then actually of *retarded type*.

Comparing this result with Proposition 3 yields

$$c<\bar{C}_{\mathrm{D}}(\vec{\tau}),\quad \forall c\in\mathbb{R}\quad\text{with } f(c;\ \vec{\tau})>1. \tag{19}$$

On the contrary, if $f(c;\ \vec{\tau})<1$ then there exists a number $c_{\mathrm{l}}<c$ such that $f(c_{\mathrm{l}};\ \vec{\tau})<1$. By the first assertion of Lemma 4 there exists a number $\varepsilon>0$ such that

$$c_{\mathrm{D}}(\vec{\tau}+\delta\vec{\tau})<c_{\mathrm{l}}<c,\ \ \forall\delta\vec{\tau}\in\mathbb{R}^m\quad\text{with}\quad\|\delta\vec{\tau}\|<\varepsilon.$$

From Proposition 3 this result implies

$$c>\bar{C}_{\mathrm{D}}(\vec{\tau}),\quad \forall c\in\mathbb{R}\quad\text{with } f(c,\ \vec{\tau})<1. \tag{20}$$

Combining (20) and (19) yields that the zero of the strictly decreasing function $f(c,\ \vec{\tau})-1$ is equal to $\bar{C}_{\mathrm{D}}(\vec{\tau})$.

The continuity of $\bar{C}_{\mathrm{D}}$ w.r.t. $\vec{\tau}$ follows from the continuity of $f(c;\ \vec{\tau})$ w.r.t. both its argument c and parameters $\vec{\tau}$. □

Remark 7. Since $f(c;\ \vec{\tau})\leqslant\sum_{k=1}^{m}\|H_k\|\mathrm{e}^{-c\tau_k}$ an upper bound on $\bar{C}_{\mathrm{D}}(\vec{\tau})$ is given by the unique solution of

$$\sum_{k=1}^{m}\|H_k\|\mathrm{e}^{-c\tau_k}-1=0 \tag{21}$$

while $\sum_{k=1}^{m}\|H_k\|<1$ is a sufficient condition for strong stability.

To conclude, we mention some special cases where the above expressions become very simple. In case of one delay ($m=1$) we have $\gamma_0=r_\sigma(H_1)$,

$$f(c;\ \tau_1)=r_\sigma(H_1)\mathrm{e}^{-c\tau_1},\quad \bar{C}_{\mathrm{D}}(\tau_1)=\frac{1}{\tau_1}\log r_\sigma(H_1).$$

When the equation is scalar ($n=1$) we have

$$\gamma_0=\sum_{k=1}^{m}|H_k|,\quad f(c;\ \vec{\tau})=\sum_{k=1}^{m}|H_k|\mathrm{e}^{-c\tau_k}.$$

3.3. Relation with the spectrum of the neutral equation

From (10) not only the difference equation (6), yet *also* the neutral equation (1)–(2) has characteristic roots with real part *arbitrarily close* to $\bar{C}_{\mathrm{D}}(\vec{\tau})$ for certain (arbitrarily small) delay perturbations. Furthermore, this property can *not* be changed by the control law (2), since the latter does not affect the difference equation. Therefore, in any pole placement based stabilization strategy one can concentrate on the characteristic roots with real part larger than $\bar{C}_{\mathrm{D}}(\vec{\tau})$.

From the fact that the operator $T(1)$, defined in (5), only has *point spectrum* in the set

$$\{\lambda\in\mathbb{C}:\ |\lambda|>r_{\mathrm{e}}(T(1))=r_\sigma(T_{\mathrm{D}}(1))\}$$

see Hale and Verduyn Lunel (2002a), it follows that all the characteristic roots of (1)–(2) in the half plane $\Re(\lambda)\geqslant\bar{C}_{\mathrm{D}}(\vec{\tau})+\varepsilon$, $\varepsilon>0$, lie in a *compact set* and that the number of these roots (multiplicity taken into account) is *finite*.

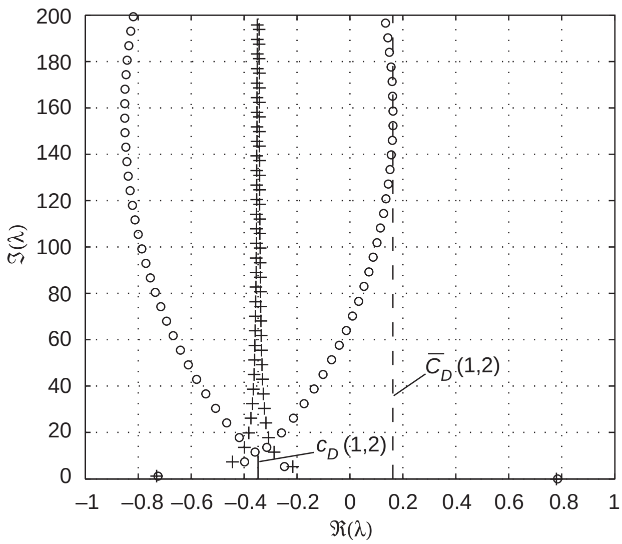

Fig. 1. Characteristic roots of (22) for delay values $\vec{\tau}=(1,2)$ ('+') and $\vec{\tau}=(0.99,2)$ ('∘'). The bound $c_{\mathrm{D}}(1,2)$ and $\bar{C}_{\mathrm{D}}(1,2)$ (the latter taking delay perturbations into account) on the real parts of the characteristic roots of the associated difference equation are also indicated.

Example 8. The difference equation associated with the neutral system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)-\frac{3}{4}x(t-\tau_1)+\frac{1}{2}x(t-\tau_2)\right) = \frac{1}{4}x(t)+\frac{3}{4}x(t-\tau_1), \tag{22}$$

where $\vec{\tau}=(1,2)$, is exponentially stable but not strongly exponentially stable because $\gamma_0=|\frac{3}{4}|+|\frac{1}{2}|>1$. Furthermore, we have $c_{\mathrm{D}}(1,2)\approx -0.3466$, which is strictly smaller than $\bar{C}_{\mathrm{D}}(1,2)\approx 0.1616$. This illustrates the non-continuity of $c_{\mathrm{D}}(\vec{\tau})$ w.r.t. $\vec{\tau}$. In Fig. 1 these values, as well as the rightmost characteristic roots of (22) are displayed (the latter indicated with '+'). To illustrate the effect of small delay perturbations, also the characteristic roots for delays values $\vec{\tau}=(0.99,2)$ are shown (indicated with '∘'). In the half plane $\Re(\lambda)\geqslant \bar{C}_{\mathrm{D}}+\varepsilon$, with $\varepsilon>0$ small, there is only one characteristic root.

4. Description of the continuous pole placement algorithm

Armed with the results of the previous section, we are ready to tackle the stabilization problem. More precisely we wish to find a gain value K such that (1)–(2) is exponentially stable and remains so when subjected to small delay perturbations. From the properties described in Section 3.3, this corresponds to having all characteristic roots in the open left half plane and the associated difference equation being strongly exponentially stable.

The main steps of the continuous pole placement method are described in Algorithm 1.

Algorithm 1. The continuous pole placement method for neutral systems:

A. Determine whether the difference equation is strongly stable by applying Proposition 1. If not, give up. If yes, compute the spectral upper bound $\bar{C}_{\mathrm{D}}(\vec{\tau})$ by applying Theorem 6. The properties of $f(c;\ \vec{\tau})$ make a bisection algorithm appropriate to find its zero.
B. Initialize $q=1$ and choose $\varepsilon>0$ small.
C. Compute the characteristic roots with $\Re(\lambda)\geqslant \bar{C}_{\mathrm{D}}(\vec{\tau})+\varepsilon$. This can be done by discretizing the solution operator $T(t)$, as illustrated in Engelborghs and Roose (1998) for the linearized solution operator around a periodic solution. Another possibility consists of applying the general purpose quasipolynomial mapping based technique of Vyhlídal and Zítek (2003).
D. Compute the sensitivity of the q rightmost characteristic roots w.r.t. changes in the feedback gain K.
E. Move the q rightmost characteristic roots in the direction of the left half plane by applying a small change to the feedback gain K, using the computed sensitivities.
F. Monitor the uncontrolled characteristic roots with $\Re(\lambda)>\bar{C}_{\mathrm{D}}(\vec{\tau})+\varepsilon$. If necessary, increase the number of controlled characteristic roots q. Stop when stability is reached or when the available degrees of freedom in the controller do not allow $\sup\Re(\lambda)$ to be further reduced or when the leftmost from the controlled characteristic roots reaches the upper bound $\bar{C}_{\mathrm{D}}(\vec{\tau})$. In the other case, go to step C.

Compared to the retarded case (Michiels et al., 2002b), the main difference is the preliminary step, analyzing the stability of the associated difference equation and computing the safe upper bound on the real parts of its characteristic roots, $\bar{C}_{\mathrm{D}}(\vec{\tau})$. This is necessary because

(1) only neutral systems with $\bar{C}_{\mathrm{D}}(\vec{\tau})<0$ (which is equivalent to the strong stability condition, see Proposition 3.1) can be stabilized safely (i.e. with the achieved stability insensitive to small delay variations).
(2) when $\bar{C}_{\mathrm{D}}(\vec{\tau})>0$, instability may be caused by characteristic roots with very large imaginary parts, which easily remain undetected when only computing a *finite* number of characteristic roots (inherent to any numerical scheme). In other words, making stability assertions without the explicit computation of $\bar{C}_{\mathrm{D}}(\vec{\tau})$ is unsafe. See Fig. 1 for an example of a root chain becoming only unstable at very high frequencies—notice the difference in scaling of both axes.
(3) the knowledge of $\bar{C}_{\mathrm{D}}(\vec{\tau})$ prevents to spend much control action to characteristic roots with smaller real part, being useless from a stabilization point of view.

For details regarding the implementation of Steps B–F, being analogous to the retarded case, practical aspects and convergence properties of the iterative scheme we refer to

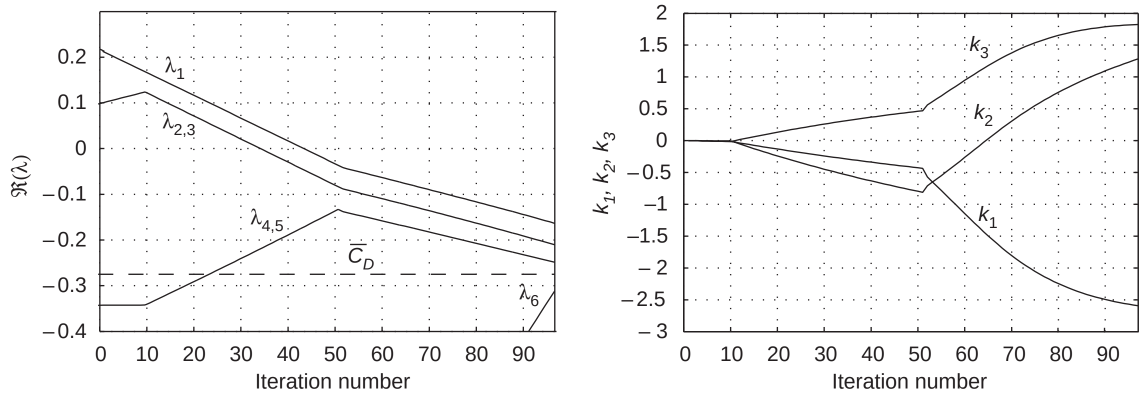

Fig. 2. Results of the continuous pole placement procedure applied to system (23). (Left) Real parts of the controlled characteristic roots. (Right) Coefficients of the feedback gain.

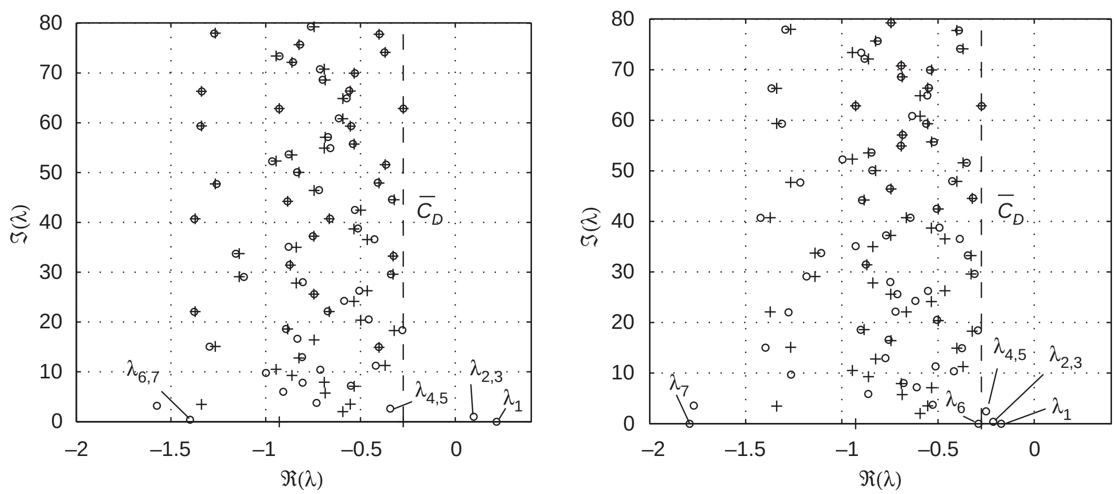

Fig. 3. (Left) Spectrum of neutral system (23) for $u=0$. (Right) Spectrum of stabilized neutral system (23). +—spectrum of difference equation associated to (23).

Michiels et al. (2002b). A numerical example is presented in the next section.

5. Illustrative example

Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(x(t) - H_1 x(t-\tau_1) - H_2 x(t-\tau_2)) \\ = A\ x(t) + Bu(t-0.5), \tag{23}$$

where $u \in \mathbb{R}$,

$$H_1 = \begin{bmatrix} 0 & 0.2 & -0.4 \\ -0.5 & 0.3 & 0 \\ 0.2 & 0.7 & 0 \end{bmatrix},\ H_2 = \begin{bmatrix} -0.3 & -0.1 & 0 \\ 0 & 0.2 & 0 \\ 0.1 & 0 & 0.4 \end{bmatrix},$$

$$A = \begin{bmatrix} -4.8 & 4.7 & 3 \\ 0.1 & 1.4 & -0.4 \\ 0.7 & 3.1 & -1.5 \end{bmatrix},\ B = \begin{bmatrix} 0.3 \\ 0.7 \\ 0.1 \end{bmatrix},$$

$\vec{\tau} = (0.7,\ 1.7)$.

The neutral system is unstable, because it has one real characteristic root $\lambda_1 \approx 0.2180$ and one pair of complex conjugate characteristic roots $\lambda_{2,3} \approx 0.0976 \pm 1.0396i$, located to the right of the stability boundary.

Performing step A of Algorithm 1 yields $\gamma_0 = 0.7507$, i.e. the difference equation is strongly stable, and $\bar{C}_{\mathrm{D}} = -0.2751$.

By applying steps B–F of Algorithm 1 we may attempt to stabilize the neutral system. Since the feedback gain has three parameters, whose initial values are $K = [k_1\ k_2\ k_3] = [0\ 0\ 0]$, up to three characteristic roots (couples of characteristic roots if they are complex) can be controlled, i.e. continuously shifted to the left. The results of iterations of the continuous pole placement algorithm are shown in Fig. 2. First, only the real characteristic root λ_1 is being shifted to the left until it gets close to the real part of the pair $\lambda_{2,3}$. Then, from iteration number 10 on, also the pair is being shifted. In order to ensure a numerically stable computation, the distances in real parts of the characteristic roots are kept larger than a default value. At iteration number 51, the couple $\lambda_{4,5}$ is attached to the group of the controlled characteristic roots. Since no more characteristic roots can be controlled by three parameters of the feedback gain, the procedure terminates at iteration 98 when characteristic root λ_6 gets close to $\lambda_{4,5}$. Using the resulting feedback gain $K = [-2.593\ 1.284\ 1.826]$, the system has been stabilized.

To illustrate the impact of the stabilizing feedback gain on the system dynamics, we show in Fig. 3 the spectra of the uncontrolled system and the system stabilized with the feedback gain resulting from the continuous pole placement procedure.

6. Conclusions

Spectral properties of neutral equations were derived, laying the basis for a practical stabilization algorithm, whose applicability was illustrated with an example.

Acknowledgements

This note presents results of the Project IUAP P5, funded by the Interuniversity Poles of Attraction—Belgian Federal Science Policy Office. Wim Michiels is a post-doctoral fellow of the Fund for Scientific Research—Flanders (Belgium). The present research has been supported by the Ministry of Education of the Czech Republic, Project IM6840770004.

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