Computation of stability regions in parameter spaces

Consider the system

$$\dot{x}(t) = A_0(p)x(t) + \sum_{i=1}^{m} A_i(p)x(t - \tau_i(p)) \quad (1)$$

under appropriate initial conditions, with

$$p := (p_1, \ldots, p_{n_p}) \in \mathcal{D} \subset \mathbb{R}^{n_p}$$

the parameters. We assume that

$\tau_i(p) \geq 0,\ \forall p \in \mathcal{D},\ \ p \mapsto A_i(p),\ p \mapsto \tau_i(p)$: smooth

The *characteristic matrix* of (1) is given by

$$M(\lambda;\ p) := \lambda I - A_0(p) - \sum_{i=1}^{m} A_i(p)e^{-\lambda\tau_i(p)},$$

and the *characteristic function* by

$$H(\lambda;\ p) := \det\left(\lambda I - A_0(p) - \sum_{i=1}^{m} A_i(p)e^{-\lambda\tau_i(p)}\right).$$

Definitions

(1) The set of values $p \in \mathcal{D}$ for which the zero solution of (1) is exponentially stable is called the *stability domain* or *stability region*.

(2) The *stability crossing boundary* of the delay system (1) is the set of points $p \in \mathcal{D}$ for which the characteristic function $H(\lambda;\ p)$ has at least one zero on the imaginary axis.

If $n_p = 2$, then the stability crossing boundary typically consists of a number of curves, the *so-called stability crossing curves*.

Basic idea to compute stability regions

1. Solve the equation

   $$H(j\omega;\ p) = 0, \quad \omega \geq 0,$$

   for p.
   - analytically (simple cases)
   - numerical continuation

   This yields a *partition* of the parameter space in regions where the number of unstable characteristic roots is constant.

2. Compute the number of unstable characteristic roots in each region:
   - by computing the rightmost characteristic roots (one point per region)
   - by characterizing the crossing direction of the characteristic roots

Theoretical example

$$\dot{x}(t) = -kx(t-\tau), \quad k \geq 0,\ \tau \geq 0$$

Stability crossing boundary

$$H(\lambda;\ \tau, k) = \lambda + ke^{-\lambda\tau}$$

$$H(0;\ \tau, k) = 0 \Leftrightarrow k = 0$$

$$\begin{aligned} H(j\omega;\ \tau, k) = 0,\ \omega > 0 \quad & \Leftrightarrow j\omega + ke^{-j\omega\tau} = 0 \\ & \Leftrightarrow \left\{ \begin{array}{l} k\cos\omega\tau = 0 \\ k\sin\omega\tau = \omega \end{array} \right. \\ & \Leftrightarrow \left\{ \begin{array}{l} \omega\tau = \frac{\pi}{2} + 2\pi l,\ \ l = 0, 1, \ldots \\ \omega = k \end{array} \right. \\ & \\ & \Leftrightarrow k\tau = \tfrac{\pi}{2} + 2\pi l,\ \ l = 0, 1, \ldots \end{aligned}$$

Crossing direction w.r.t. a change of k

an isolated characteristic root defines (locally) a function $k \mapsto \lambda(k)$, satisfying

$$\lambda(k) + ke^{-\lambda(k)\tau}$$

Taking the derivative w.r.t. k yields:

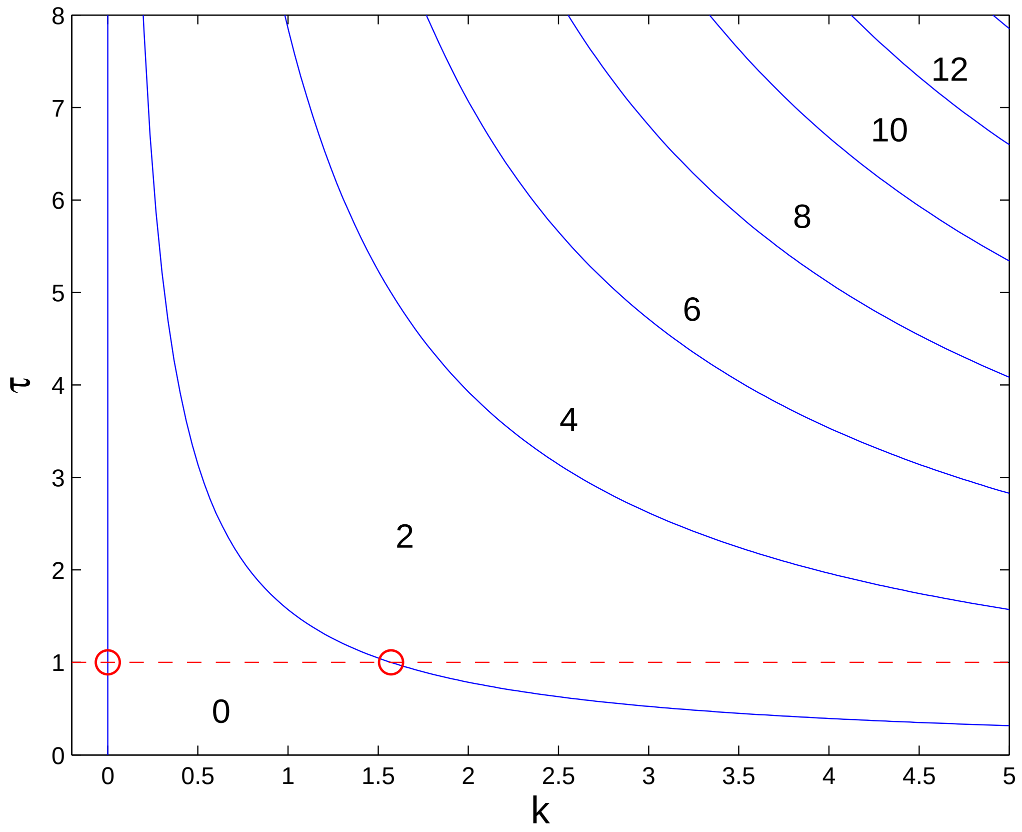
$$\frac{d\lambda(k)}{dk} = -\frac{e^{-\lambda\tau}}{1 - k\tau e^{-\lambda\tau}}$$

$$\Rightarrow \left.\frac{d\lambda(k)}{dk}\right|_{\lambda=0,k=0} = -1$$

For $k \neq 0$:

$$\begin{aligned}\left.\tfrac{d\lambda(k)}{dk}\right|_{\lambda=j\omega} &= -\tfrac{e^{-j\omega\tau}}{1-k\tau e^{-j\omega\tau}} \\ &= \tfrac{j\omega/k}{(1+j\omega\tau)} \\ &= \tfrac{\omega^2\tau+j\omega}{k(1+(\omega\tau)^2)}\end{aligned}$$

$$\Rightarrow \left.\frac{d\Re(\lambda(k))}{dk}\right|_{\lambda=j\omega} = \frac{\omega^2\tau}{k(1+(\omega\tau)^2)} > 0$$



Remark:

$$\lambda + ke^{-\lambda\tau} = 0$$
$$\Leftrightarrow$$
$$\tilde{\lambda} + \tilde{k}e^{-\tilde{\lambda}} = 0,$$

where $\tilde{\lambda} = \lambda\tau$ and $\tilde{k} = k\tau$. As this scaling does not affect stability it is in fact sufficient to look at a one-parameter problem.

Example from a congestion control application

See model pp.266-268

$$\ddot{q}(t) + \frac{1}{c}\dot{q}(t) + \frac{1}{c}\dot{q}(t-1) + \frac{kc^2}{2}q(t-1) = 0,$$

with 2 parameters: $c > 0$ and $k \in \mathbb{R}$.

$$H(\lambda;\ k, c) := \lambda^2 + \frac{1}{c}\lambda + \frac{1}{c}\lambda e^{-\lambda} + \frac{kc^2}{2}e^{-\lambda}.$$

For $k = 0$ we get

$$\lambda\left(\lambda + \frac{1}{c} + \frac{1}{c}e^{-\lambda}\right) = 0:$$

- one zero characteristic root
- all other characteristic roots are in $\mathbb{C}^-$

Stability crossing boundary

$$\lambda = 0 \Leftrightarrow k = 0$$

$$\lambda = j\omega,\ \omega > 0 \Leftrightarrow \begin{cases} c = \frac{1+\cos\omega}{\omega\sin\omega} \\ k = \frac{2\omega^4(\sin\omega)^2}{(1+\cos\omega)^2} \end{cases}$$

Crossing direction w.r.t. an increase of k

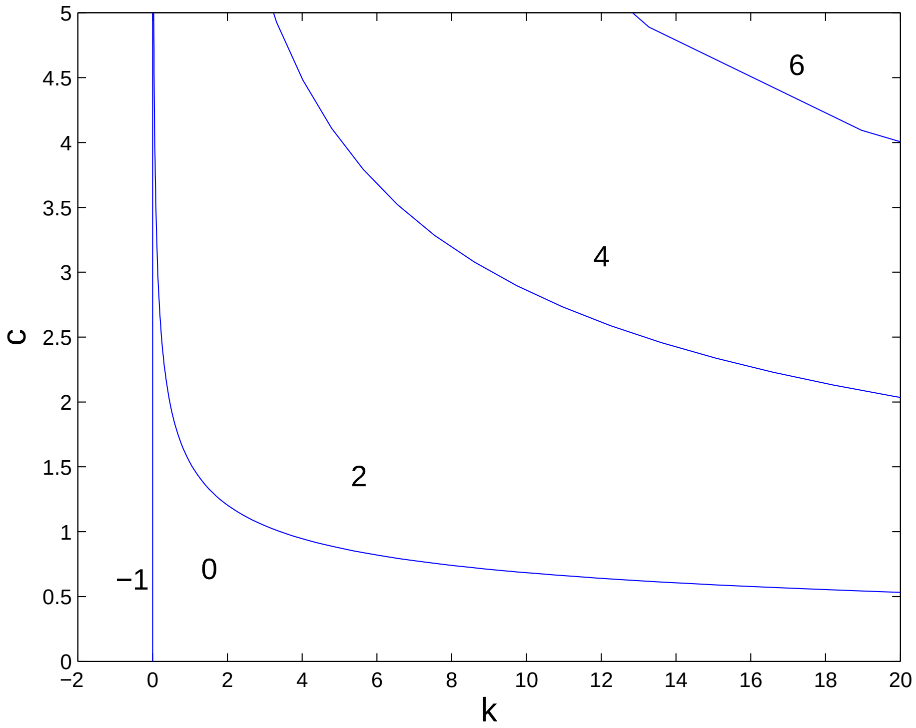
$$\left.\frac{d(\lambda)}{dk}\right|_{\lambda=0,k=0} = -\frac{c^3}{4} < 0$$

$\longrightarrow$ no stability region for $k < 0$

$$\left.\frac{d\Re(\lambda)}{dk}\right|_{\lambda=j\omega} = \frac{\omega^2 c^2 \left(\frac{1}{2c}\omega^2 + \frac{kc^2}{4}\omega^2 + \frac{k}{4} + \frac{kc}{4}\right)}{\left(\frac{\omega^2}{c} - \frac{kc}{2} + \frac{\omega^2}{c^2} + \frac{kc^2\omega^2}{2}\right)^2 + \left(\frac{\omega^3}{c} - \omega k c^2 - \frac{kc\omega}{2}\right)^2}$$

positive if k is positive !

$\longrightarrow$ only one (connected) stability region



General formula for the crossing direction

Characteristic root with multiplicity one:

Scalar (characteristic) equation

$$H(\lambda(p);\ p) = 0.$$

$$\rightarrow \frac{\partial \lambda}{\partial p_i} = -\frac{\dfrac{\partial}{\partial p_i} H(\lambda;\ p)}{\dfrac{\partial}{\partial \lambda} H(\lambda;\ p)}$$

Equation in matrix form

$$M(\lambda(p);\ p)\ u(p) = 0, \qquad u(p) \in \mathbb{C}^{n\times 1} \setminus \{\vec{0}\}$$

There always exists a vector $v \in \mathbb{C}^{n\times 1} \setminus \{\vec{0}\}$ such that

$$v^*(p)\ M(\lambda(p);\ p) = 0.$$

$$\rightarrow \frac{\partial \lambda(p)}{\partial p_i} = -\frac{v^*(p) \dfrac{\partial}{\partial p_i} M(\lambda;\ p)\ u(p)}{v^*(p) \dfrac{\partial}{\partial \lambda} M(\lambda;\ p)\ u(p)}$$

Thus: explicitly computing the characteristic equation (→ expanding a determinant) is not necessary

Remark on non-simple characteristic roots

semisimple characteristic roots
(same algebraic and geometric multiplicity):

→ same behavior as simple characteristic roots

non-semisimple characteristic roots
(algebraic multiplicity higher than geometric multiplicity):

→ functions $p \mapsto \lambda_i(p)$ in general continuous but not differentiable
Example:

$$\lambda^2 - p = 0,\ p \in \mathbb{R}$$
$$\to \lambda = \begin{cases} \pm\sqrt{p}, & p \geq 0, \\ \pm\sqrt{-p}\ j, & p < 0. \end{cases}$$

→ higher order analysis necessary

Numerical continuation

Automatic computation of stability crossing curves

Characteristic matrix $M(\lambda;\ p) \in \mathbb{C}^{n\times n}$

A characteristic root on the imaginary axis is described by the following determining system:

$$\left\{\begin{array}{l}\Re(M(j\omega;\ p)v) = 0\\ \Im(M(j\omega;\ p)v) = 0\\ \Re(n(v)) = 0\\ \Im(n(v)) = 0\end{array}\right. , \qquad (1)$$

where $n(v)$ is a normalization constraint.

Two free parameters, $p = (p_1, p_2)$:
- $2n + 2$ equations
- $2n + 3$ (real) unknowns: $v \in \mathbb{C}^{n\times 1}$, $\omega \in \mathbb{R}$, $p \in \mathbb{R}^2$.

→ solutions of (1) generically define curves
→ stability crossing curves = projection on the (p_1, p_2)-space

Predictor-corrector method:

Given a (some) computed point(s) on the curve, repeat:

1. predict a new point on the curve
   e.g. using linear extrapolation
   (secant predictor, tangent predictor,. . .)

$$\begin{aligned}
\widehat{p}_1^{(j+1)} &= p_1^{(j)} + \tfrac{\epsilon^{(j)}}{\|p^{(j)}-p^{(j-1)}\|_2}\ (p_1^{(j)} - p_1^{(j-1)}) \\
\widehat{p}_2^{(j+1)} &= p_2^{(j})) + \tfrac{\epsilon^{(j)}}{\|p^{(j)}-p^{(j-1)}\|_2}\ (p_2^{(j)} - p_2^{(j-1)}) \\
\widehat{\omega}^{(j+1)} &= \omega^{(j)} + \tfrac{\epsilon^{(j)}}{\|p^{(j)}-p^{(j-1)}\|_2}\ (\omega^{(j)} - \omega^{(j-1)}) \\
\widehat{v}^{(j+1)} &= v^{(j)} + \tfrac{\epsilon^{(j)}}{\|p^{(j)}-p^{(j-1)}\|_2}\ (v^{(j)} - v^{(j-1)})
\end{aligned}$$

   $\epsilon^{(j)}$: steplength in the j-the step

2. compute a new point on the curve
   using Newton’s methods on
   (1) + additional condition
   (arclength, pseudo-arclength, ...)

$$\left(p_1 - p_1^{(j)}\right)^2 + \left(p_2 - p_2^{(j)}\right)^2 = \left(\epsilon^{(j)}\right)^2, \text{ or}$$

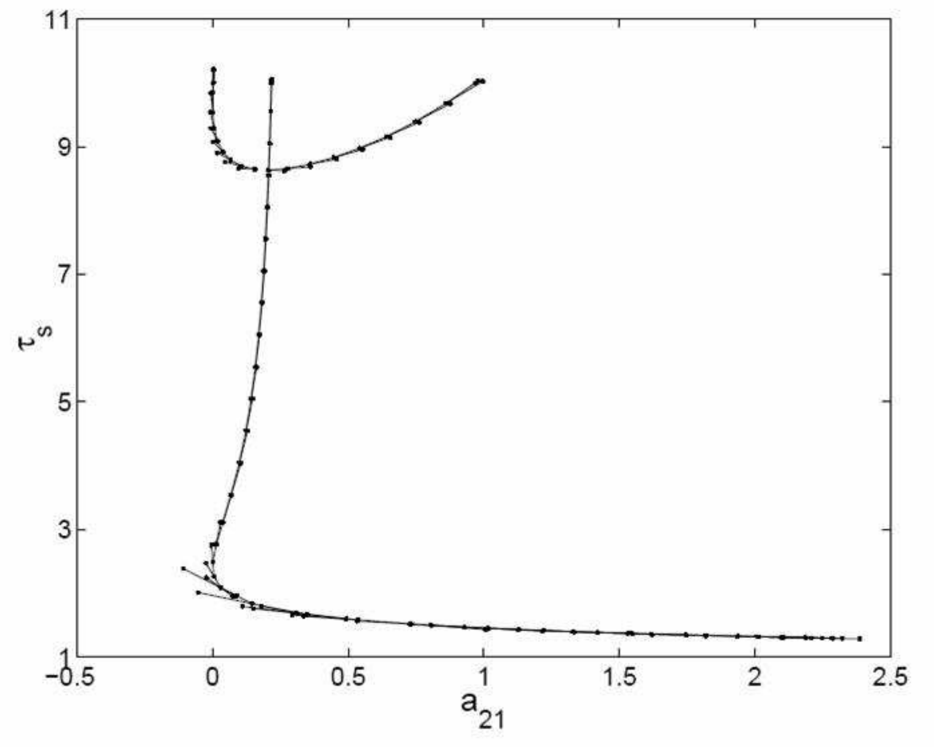
$$\left\langle p^{(j)} - p^{(j-1)}, p - \widehat{p}^{(j+1)}\right\rangle = 0.$$

Some features:

- steplength can be automatically adapted based
  on # Newton iterations needed to reduce residual
- turning point can be passed automatically

- starting point: branch in a one parameter space

  $+$ rightmost characteristic roots computations

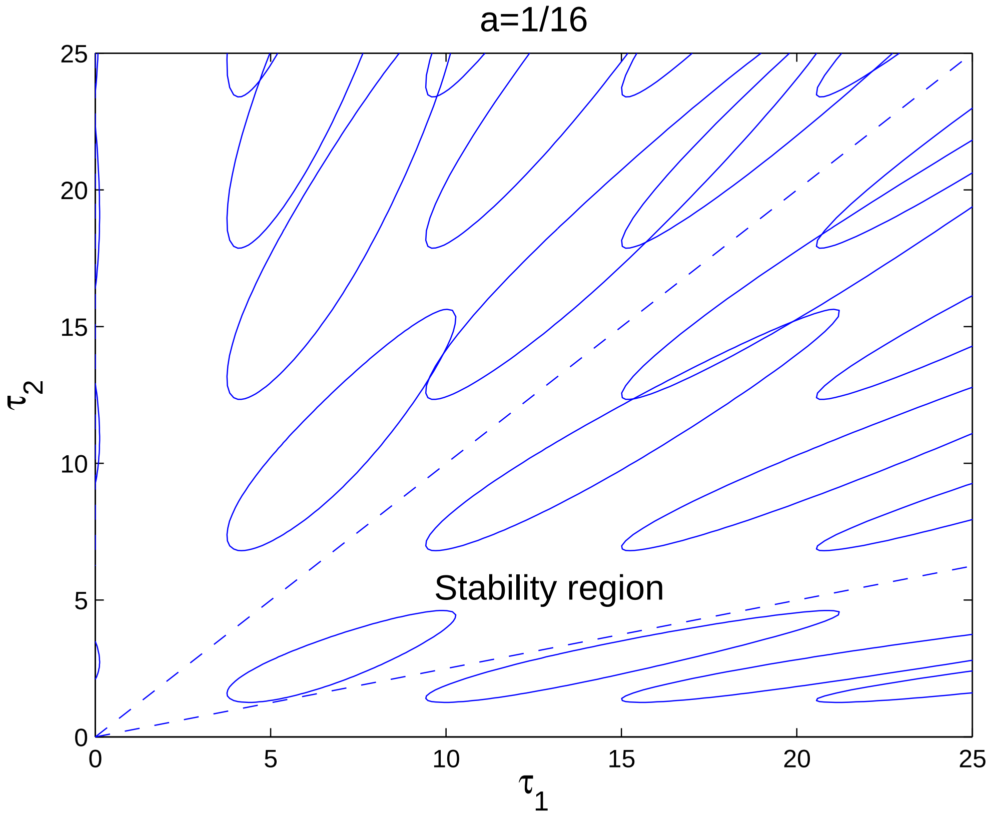


Computation of stability regions in delay parameter spaces

Example:

Stability regions in a two parameter space

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau_1) + A_2 x(t-\tau_2)$$



Invariance properties

Roots on the imaginary axis

For $\omega \neq 0$, we have:

$$
\begin{array}{l}
H(j\omega;\ \tau_1, \tau_2, \ldots) = 0 \\
\Leftrightarrow H\left(j\omega;\ \tau_1 + l_1 \frac{2\pi}{\omega}, \tau_2 + l_2 \frac{2\pi}{\omega}, \ldots\right) = 0,\ l_1, l_2, \ldots \in \mathbb{Z}
\end{array}
$$

Proof: equation in independent variables

$$\omega,\ e^{-j\omega\tau_1}, e^{-j\omega\tau_2}, \ldots$$

Remark: shift is frequency dependent !

Crossing direction of a simple root on the imaginary axis

The quantity

$$\text{sign } \left.\frac{\partial \Re(\lambda)}{\partial \tau_k}\right|_{\lambda=j\omega,\ \omega>0}$$

is invariant w.r.t. shifts of $2\pi l/\omega,\ l \in \mathbb{Z}$, in the direction of the τ_k axis.

Similar results hold if the delays are restricted. Example:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - i\tau),\ \ \tau > 0$$

→ the delays are multiples of the same number or *commensurate*.

Let $H(\lambda;\ \tau)$ and $M(\lambda;\ \tau)$ be the characteristic function and characteristic matrix. We have:

1. Invariance w.r.t. delay shifts

$$\begin{array}{l} H(j\omega;\ \tau) = 0 \\ \Leftrightarrow H\left(j\omega;\ \tau + l\frac{2\pi}{\omega}\right) = 0,\ \ l \in \mathbb{Z} \end{array}$$

2. Crossing direction

$$\text{sign}\ \left.\frac{\partial \Re(\lambda)}{\partial \tau}\right|_{\lambda = j\omega,\ \omega > 0}$$

   is invariant w.r.t. delay shifts of $2\pi l/\omega,\ \ l \in \mathbb{Z}$.

   No corollary of result of previous slide, follows from

$$\Re\left[\frac{\partial \lambda}{\partial \tau}\right]^{-1} = -\Re \frac{v^* u}{j\omega v^* \sum_{i=1}^{m} i A_i u e^{-j\omega\tau i}},$$

   with $v^*(u)$ left (right) eigenvector of $M(j\omega;\ \tau)$

Algebraic methods

We discuss systems with one delay,

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau),$$

but the results trivially extend to systems with multiple commensurate delays

Principles:

1. The equation

   $$\det(j\omega I - A_0 - A_1 e^{-j\omega\tau}) = 0,$$

   with ω, τ free parameters, can be seen as an equation in two independent variables:

   $\omega \in \mathbb{R}$
   $z = e^{-j\omega\tau} \in \mathcal{C}(0,1)$ (unit circle in $\mathbb{C}$)
   $\rightarrow$ analyze

   $$\det(j\omega I - A_0 - A_1 z) = 0$$

2. *Eliminate* one of the two variables and solve the equation
3. given ω and z compute the corresponding delay values, via $z = e^{-j\omega\tau}$

Matrix pencil technique

= one way to eliminate ω

$$\det(j\omega I - A_0 - A_1 z) = 0$$

is equivalent with (notation $\sigma(\cdot)$: spectrum)

$$j\omega \in \sigma(A_0 + A_1 z)$$

$$-j\omega \in \sigma\left(A_0^T + A_1^T \bar{z}\right)$$

It follows that

$$0 \in \sigma\left((A_0 + A_1 z) \oplus \left(A_0^T + A_1^T \bar{z}\right)\right),$$

or, equivalently,

$$\det\left((A_0 + A_1 z) \oplus \left(A_0^T + A_1^T \bar{z}\right)\right) = 0$$

This is a generalized eigenvalue problem, which can be transformed to

$$\det\left(z \begin{bmatrix} I & 0 \\ 0 & A_1 \otimes I \end{bmatrix} + \begin{bmatrix} 0 & I \\ I \otimes A_1^T & A_0 \oplus A_0^T \end{bmatrix}\right) = 0$$

Theorem:

Assume that the matrix pencil

$$\Lambda = z \begin{bmatrix} I & 0 \\ 0 & A_1 \otimes I \end{bmatrix} + \begin{bmatrix} 0 & I \\ I \otimes A_1^T & A_0 \oplus A_0^T \end{bmatrix}$$

is regular.

The characteristic equation has a root $j\omega_0$, $\omega_0 > 0$, for some positive delay value τ if and only if there exists a complex number

$$z_0 \in \sigma(\Lambda) \cap \mathcal{C}(0,1)$$

such that

$$j\omega_0 \in \sigma\left(A_0 + A_1 z_0\right).$$

Furthermore, the corresponding delay values are given by

$$\mathcal{T}_{\omega_0} = \left\{ \tfrac{\angle \overline{z}_0}{\omega_0} + \tfrac{2\pi\ell}{\omega_0} > 0 : \ z_0 \in \sigma(\Lambda) \cap \mathcal{C}(0,1), \right. \\ \left. j\omega_0 \in \sigma\left(A_0 + A_1 z_0\right), \ \ell \in \mathbb{Z} \right\}.$$

If the root is simple and the delay is increased, then the crossing is towards instability (stability) if

$$\Re\left\{ \frac{\omega_0}{j v_0^* u_0} z_0 v_0^* A_1 u_0 \right\} > 0 \quad (< 0),$$

with $(j\omega_0 I - A_0 - A_1 z_0)u_0 = 0, \quad v_0^*(j\omega_0 I - A_0 - A_1 z_0) = 0$

Example of resulting algorithm:

$\sigma(\Lambda) \cap \mathcal{C}(0,1)$	freq.	critical delays	crossing direction
z_1	ω_1	$\tau_{1,1} + \frac{2\pi l}{\omega_1},\ l \in \mathbb{Z}$	$+$
	ω_2	$\tau_{1,2} + \frac{2\pi l}{\omega_2},\ l \in \mathbb{Z}$	$-$
z_2	ω_3	$\tau_{2,3} + \frac{2\pi l}{\omega_3},\ l \in \mathbb{Z}$	$+$
z_3	ω_4	$\tau_{3,4} + \frac{2\pi l}{\omega_4},\ l \in \mathbb{Z}$	$-$
	ω_5	$\tau_{3,5} + \frac{2\pi l}{\omega_5},\ l \in \mathbb{Z}$	$+$

Notice:

1. *finite number of elements* in the first two columns.
   Computing the stability crossing boundary (here: critical delay values) is essentially a finite-dimensional problem.

2. Crossing direction only depends on the two independent variables z and $j\omega$ in

$$\det(j\omega I - A_0 - A_1 z) = 0.$$

Other elimination technique

(elimination of z)

Example: quasi-polynomials with one delay:

$$H(\lambda;\ \tau) := Q(\lambda) + P(\lambda)e^{-\lambda\tau}$$

with Q and P polynomials satisfying $\deg(Q) > \deg(P)$. The relation

$$Q(j\omega) = -P(j\omega)e^{-j\omega\tau} \tag{1}$$

implies

$$|Q(j\omega)| = |P(j\omega)| \tag{2}$$

Algorithm:

1. compute ω from (2)
2. compute the corresponding $z = e^{-j\omega\tau}$ from (1)
3. compute the critical delay values

Crossing direction for a delay increase.

If all zeros of the real function $F(\omega) := |Q(j\omega)|^2 - |P(j\omega)|^2$ are simple, then a characteristic root $j\omega_0$ crosses the imaginary axis toward instability (stability) if

$$F'(\omega_0) > 0 \quad (< 0).$$

Example (delayed output feedback, p. 241)

$$Q(\lambda) + kP(\lambda)e^{-\lambda\tau} = 0,$$

where

$$k = 0.0025, P(\lambda) = 1,$$

$$Q(\lambda) = \lambda^6 + p_1\lambda^5 + p_2\lambda^4 + p_3\lambda^3 + p_4\lambda^2 + p_5\lambda + p_6,$$

with

$$\begin{array}{llll} p_1 = & -6.00000e-04 & p_4 = & 4.34819e-01 \\ p_2 = & 1.40816e+00 & p_5 = & -8.69638e-05 \\ p_3 = & -5.63266e-04 & p_6 = & 2.66556e-02 \end{array}$$

From the relation

$$|Q(j\omega)| = |kP(j\omega)|$$

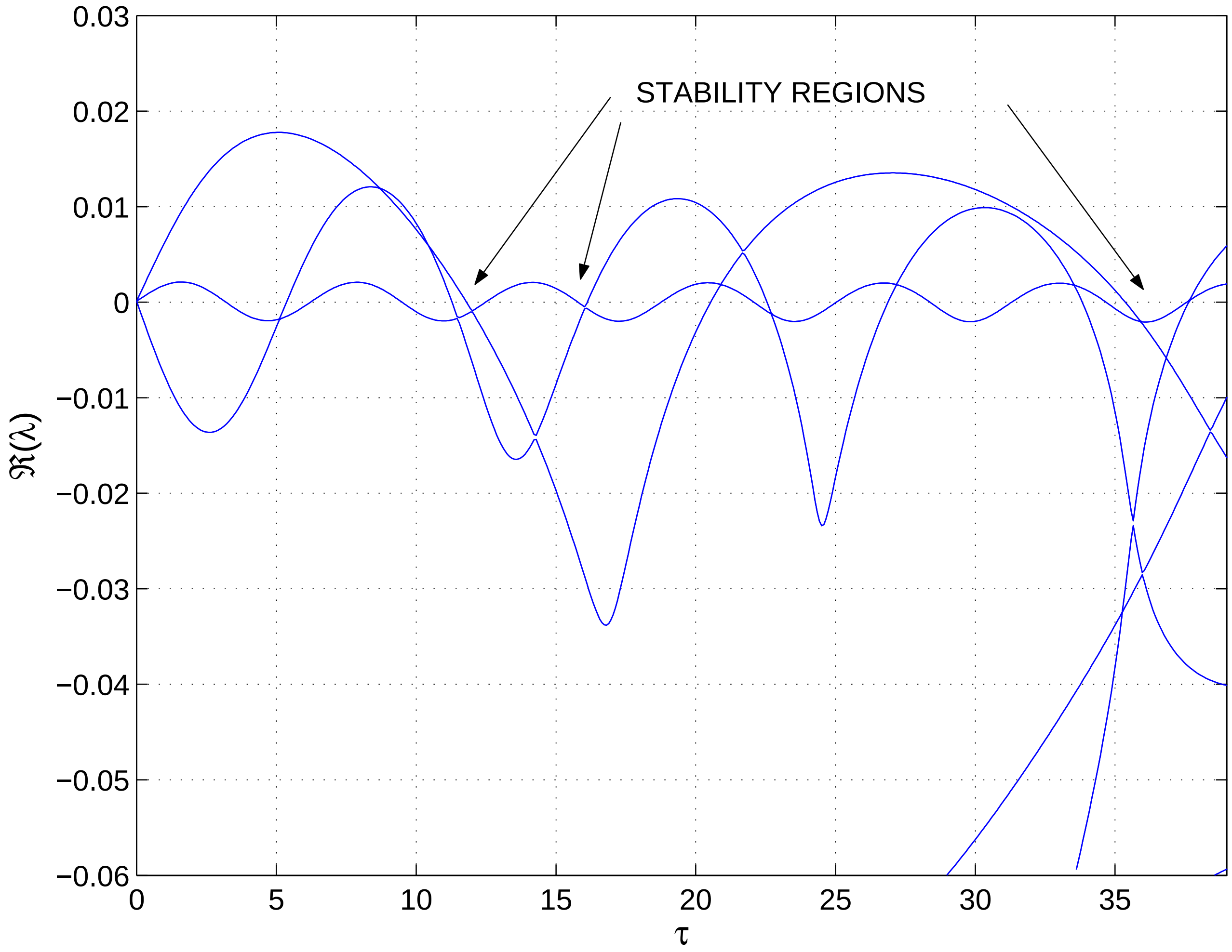
and the fact that the sign of F' alternates at successive zeros of F we obtain:

frequency	crossing direction
$\omega_1 = 1.0019959e+00$	$+$
$\omega_2 = 9.9795792e-01$	$-$
$\omega_3 = 5.8408171e-01$	$+$
$\omega_4 = 5.5740265e-01$	$-$
$\omega_5 = 3.0572050e-01$	$+$
$\omega_6 = 2.6663916e-01$	$-$

After ordering of critical delays:

Critical delays	crossing frequency	# unstable roots changes to:
		6
1.2048745e-02	ω_4	4
3.1964843e+00	ω_2	2
5.3645410e+00	ω_3	4
6.2201470e+00	ω_1	6
9.4925266e+00	ω_2	4
1.1284305e+01	ω_4	2
1.1802168e+01	ω_6	0
1.2490817e+01	ω_1	2
1.5788569e+01	ω_2	0
1.6121915e+01	ω_3	2
1.8761486e+01	ω_1	4
2.0536234e+01	ω_5	6
2.2084611e+01	ω_2	4
2.2556560e+01	ω_4	2
2.5032156e+01	ω_1	4
2.6879289e+01	ω_3	6
2.8380653e+01	ω_2	4
3.1302825e+01	ω_1	6
3.3828816e+01	ω_4	4
3.4676696e+01	ω_2	2
3.5366543e+01	ω_6	0
3.7573495e+01	ω_1	2
3.7636663e+01	$\omega 3$	4
⋮	⋮	⋮

Validation by computation of rightmost characteristic roots



Methods based on a substitution

Example

Idea:

$$\begin{array}{l} \{e^{-j\omega}:\ \omega\in\mathbb{R}\} = \mathcal{C}(0,1) \\ \left\{\frac{1-j\omega}{1+j\omega}:\ \omega\in\mathbb{R}\right\} = \mathcal{C}(0,1)\setminus\{-1\} \end{array}$$

→ Rekasius, Cayleigh transform, pseudo-delay technique

$$e^{-j\omega\tau} = \frac{1-j\omega T}{1+j\omega T}.$$

Analyzing

$$\det\left(j\omega I - A_0 - A_1 e^{-j\omega\tau}\right) = 0$$

in the (ω,τ) space

⇒ analyzing

$$\det\left(j\omega I - A_0 - A_1 \frac{1-j\omega T}{1+j\omega T}\right) = 0$$

in the (ω, T) space.

⇒ analyzing the stability boundary of a parameterized polynomial

(Routh Hurzitz, root locus, matrix pencils,. . .)

Geometric methods

→ multiple delays

Example

$H(\lambda;\ \tau_1,\tau_2) := p_0(\lambda) + p_1(\lambda)e^{-\lambda\tau_1} + p_2(\lambda)e^{-\lambda\tau_2}$

Stability regions in the parameter space (τ_1,τ_2)?

If $p_0(j\omega;\ \tau_1,\tau_2) \neq 0$ for all $\omega \geq 0$ then

$$H(j\omega;\ \tau_1,\tau_2) = 0$$

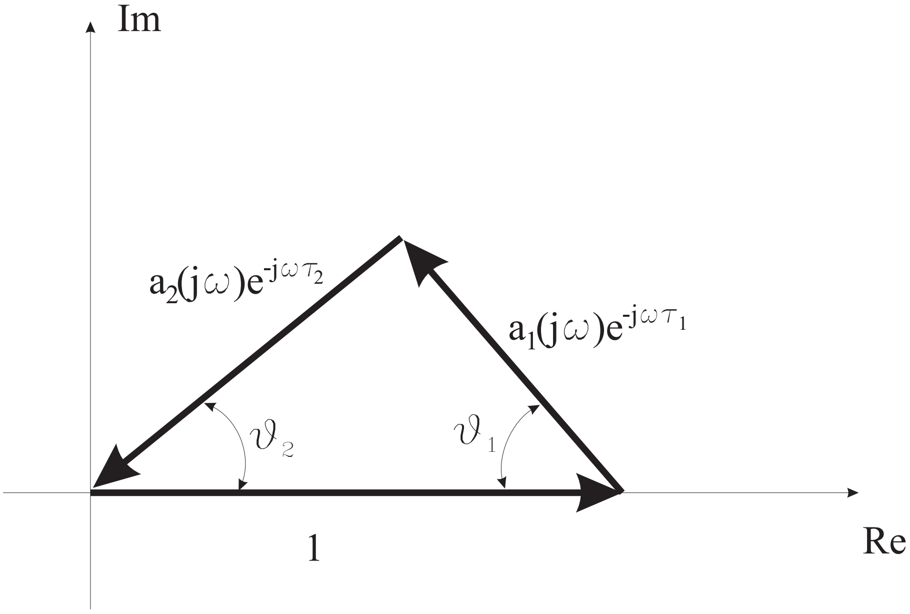
$$\Updownarrow$$

$$1 + a_1(j\omega)e^{-j\omega\tau_1} + a_2(j\omega)e^{-j\omega\tau_2} = 0 \qquad (1)$$

where

$$a_i(\lambda) = \frac{p_i(\lambda)}{p_0(\lambda)},\ \ i = 1, 2.$$

Idea: (1) is satisfied if and only if the "vectors" 1, $a_1(j\omega)e^{-j\omega\tau_1}$ and $a_2(j\omega)e^{-j\omega\tau_2}$ form a *triangle*.



Basic algorithm
(frequency sweeping test, elimination based):

- Determine the critical frequencies ω:

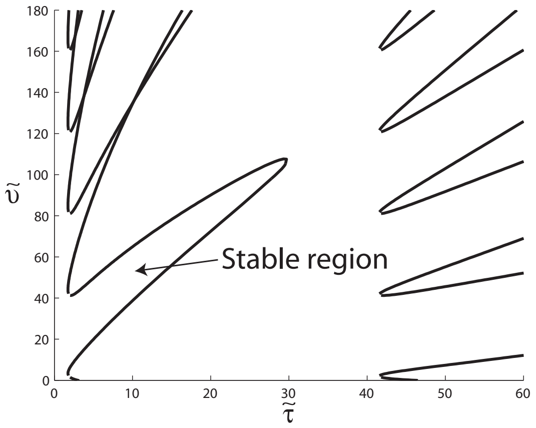
$$\begin{aligned} |a_1(j\omega)|+|a_2(j\omega)| &\geq 1, \\ -1 \leq |a_1(j\omega)|-|a_2(j\omega)| &\leq 1 \end{aligned}$$

  (condition on the length of the arrows in the triangle)

- Determine the corresponding delays (condition on the direction of the arrows)

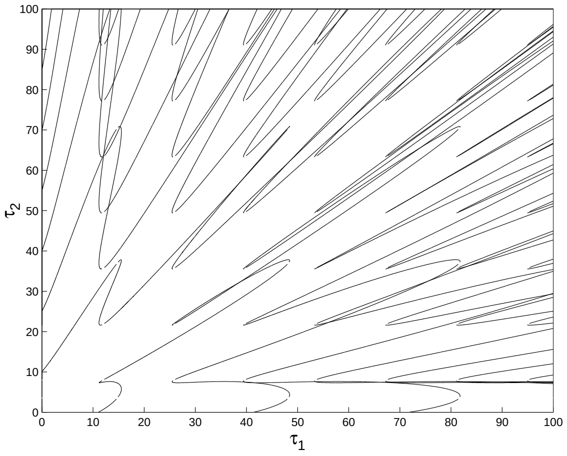
  Take into account invariance properties w.r.t. delay shifts !

Some examples of qualitative behavior:

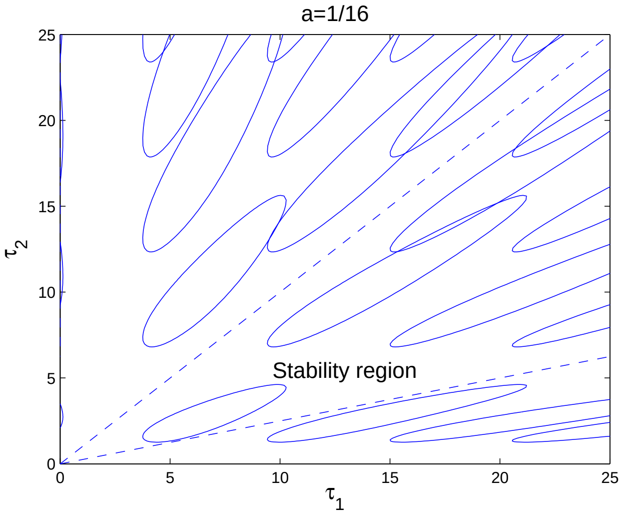


Stable region

Spiral like curves, vertically oriented



Spiral like curves, diagonally oriented



Closed curves