A General Setting for the Parametric Google Matrix

(It's not about Markov chains)

Roger Horn and Stefano Serra-Capizzano

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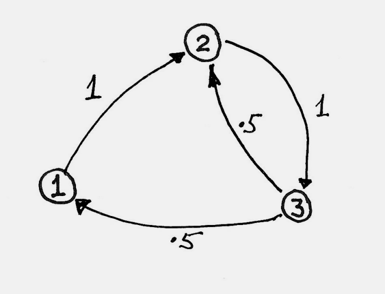
- G is nonnegative and row stochastic (need not be primitive or irreducible)
- $Ge = e$, $e^T = [1 \ \ldots \ 1]^T$

Example: Internet with three pages

.

- 

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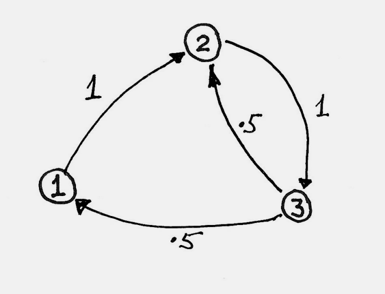
- 

- $G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ .5 & .5 & 0 \end{bmatrix}$

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- Existence OK: Perron-Frobenius

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- Even if there is a unique solution, there can be other eigenvalues of modulus one, so the power method might fail
- For the power method to work, the second-largest modulus eigenvalue of G must have modulus *strictly* less than one.

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- Every other eigenvalue of $G(c)$ has modulus at most c

PageRank reconsidered

$0 < c < 1, \quad G(c) = cG + (1-c)ev^T$

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- Popular belief: Google uses $c = .85$

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- Does something bad happen as $y \to 1$?

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- (Brauer, 1952) The eigenvalues of $A(c)$ are $\lambda, c\lambda_2, \ldots, c\lambda_n$
- If the JCF of A is

$$[\lambda] \oplus J_{n_2}(\nu_2) \oplus \cdots \oplus J_{n_k}(\nu_k),$$

and if $c \neq 0$ and $c\lambda_j \neq \lambda$ for all $j = 2, \ldots, n$, then JCF of $A(c)$ is

$$[\lambda] \oplus J_{n_2}(c\nu_2) \oplus \cdots \oplus J_{n_k}(c\nu_k).$$

Normalized left eigenvector

Suppose $0 \neq \lambda \neq c\lambda_j$, $j = 2, \ldots, n$, so λ is a simple eigenvalue of $A(c)$

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- However...if λ is not a semisimple eigenvalue of A, then there are choices of v for which $\lim_{c \to 1} y(c)$ does not exist.

Semisimple case

Suppose λ is semisimple, multiplicity m

- $S^{-1}AS = \begin{bmatrix} \lambda I_m & 0 \\ 0 & E \end{bmatrix}, S = \begin{bmatrix} X & S_2 \end{bmatrix}, (S^{-1})^* = \begin{bmatrix} Y & Z_2 \end{bmatrix},$ X, Y are n-by-m, $AX = \lambda X$, $Y^*A = \lambda Y^*$, $Y^*X = I_m$, λ is not an eigenvalue of E

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  $$y(c)^* = v^*XY^* + (1-c)\lambda v^* S_2(\lambda I_{n-m} - cE)^{-1} Z_2^*$$
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X, Y are n-by-m, $\quad AX = \lambda X, \quad Y^*A = \lambda Y^*, \quad Y^*X = I_m, \quad \lambda$ is not an eigenvalue of E
- $$y(c)^* = v^*XY^* + (1-c)\lambda v^* S_2(\lambda I_{n-m} - cE)^{-1} Z_2^*$$
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- XY^* is a uniquely defined projection (X any basis of right λ-eigenspace of A; Y the dual basis of left λ-eigenspace)
- Alternatively, $XY^* = I - (\lambda I - A)(\lambda I - A)^D$ (second form)

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- $y^{'}(1)^* = \lambda v^* S_2 (E - \lambda I)^{-1} Z_2^* = \lambda v^* (A - \lambda I)^D$
- Explicit power series for $y(c)$ around $c = 1$

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- Every eigenvalue of G of modulus 1 is semisimple

A Third Form of the Projection

- Suppose $\lambda \neq 0$ is an eigenvalue of A with maximum modulus, and that *every* eigenvalue with maximum modulus is semisimple. Then

$$XY^* = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \left(\lambda^{-1} A\right)^k$$

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- If A is a stochastic matrix, this Cesaro sum is often called the *ergodic projection*.

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- Nothing bad happens at $c = 1$

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- Matrix analysis gives a clear understanding of "pure PageRank"

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Philosophy and Computation

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- Complex algorithms involving FFT and vector extrapolation may be superior to the power method for computing (ordinary) PageRank at $c = .85$, at $c = .99$, and even in the limit as $c \to 1$.