Laguerre-Sobolev-Type Orthogonal Polynomials
A non diagonal case

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1. Goals of the presentation

Let the Sobolev-Type inner product

$$\langle p, q \rangle_S = \int_0^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx + \mathbb{P}(0)^t A \mathbb{Q}(0), \ \alpha > -1, \tag{1}$$

where p and q are polynomials with real coefficients,

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}, \ \mathbb{P}(0) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix}, \ \mathbb{Q}(0) = \begin{pmatrix} q(0) \\ q'(0) \end{pmatrix},$$

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2. The Laguerre polynomials

The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle p, q \rangle = \int_0^\infty pqx^\alpha e^{-x} dx, \;\; \alpha > -1, \;\; p, q \in \mathbb{P}. \tag{2}$$

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1 Proposition. Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials.

1. For every $n \in \mathbb{N}$,

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+1+\alpha)\, L_n^\alpha(x) + n\,(n+\alpha)\, L_{n-1}^\alpha(x), \tag{3}$$

   with $L_0^\alpha(x) = 1, L_1^\alpha(x) = x-\ (\alpha+1)$.

2. For every $n \in \mathbb{N}$,

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x)\,. \tag{4}$$

3. For every $n \in \mathbb{N}$,

$$\|L_n^\alpha\|_\alpha^2 = n!\Gamma(n+\alpha+1). \tag{5}$$

4. For every $n \in \mathbb{N}$

$$L_n^\alpha(0) = (-1)^n\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \tag{6}$$

5. For every $n \in \mathbb{N}$

$$\left(L_n^\alpha\right)'(x) = nL_{n-1}^{\alpha+1}(x). \tag{7}$$

6. For every $n \in \mathbb{N}$,

$$x\left(L_n^\alpha(x)\right)' = nL_n^\alpha(x) + n\left(n+\alpha\right)L_{n-1}^\alpha(x). \tag{8}$$

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2 Theorem. (The Mehler-Heine type formula) Let J_α be the Bessel function defined by

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j+\alpha}}{j!\Gamma(j+\alpha+1)},$$

then

$$\lim_{n\to\infty} \frac{\widehat{L}_n^\alpha(x/(n+j))}{n^\alpha} = x^{-\alpha/2} J_\alpha(2\sqrt{x}) \tag{9}$$

uniformly on compact subsets $\mathbb{C}$ and uniformly in $j \in \mathbb{N} \cup \{0\}$. Here $\widehat{L}_n^\alpha(x) = (-1)^n/n! L_n^\alpha(x)$.

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3 Theorem. If $j, l \in \mathbb{R}$, $h, k \in \mathbb{Z}$, then

$$\lim_{n\to\infty} \frac{n^{(l-j)/2} \widehat{L}_{n+k}^{\alpha+j}(x)}{\widehat{L}_{n+h}^{\alpha+l}(x)} = (-x)^{-(j-l)/2}, \tag{10}$$

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$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0), \ \alpha > -1, \tag{11}$$

where

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix},$$

$M_{0,}M_1 \geq 0$, A is a positive semidefinite matrix.

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$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} K_{n-1}(x,0) \\ K_{n-1}^{(0,1)}(x,0) \end{pmatrix}. \tag{12}$$

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From the above expression we obtain

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Thus

$$\left(\widetilde{\mathbb{L}}^\alpha(0)\right)^t \left(I + A\mathbb{K}_{n-1}(0,0)\right) = \left(\mathbb{L}^\alpha(0)\right)^t, \tag{13}$$

where

$$\mathbb{K}_{n-1}(0,0) = \begin{pmatrix} K_{n-1}(0,0) & K_{n-1}^{(1,0)}(0,0) \\ K_{n-1}^{(0,1)}(0,0) & K_{n-1}^{(1,1)}(0,0) \end{pmatrix}.$$

Notice that $I + A\mathbb{K}_{n-1}(0,0)$ is a non singular matrix.

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Notice that $I + A\mathbb{K}_{n-1}(0,0)$ is a non singular matrix. Then

$$\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha}(x) - \mathbb{L}_n^{\alpha}(0)^t \left(I + A\mathbb{K}_{n-1}(0,0)\right)^{-1} A\mathbb{K}_{n-1}(x,0).$$

On the other hand

$$I + A\mathbb{K}_{n-1}(0,0) =$$

$$\begin{aligned} &= K_{n-1}(0,0)\left[\begin{pmatrix} \frac{1}{K_{n-1}(0,0)} & 0 \\ 0 & \frac{1}{K_{n-1}(0,0)} \end{pmatrix} + A\begin{pmatrix} 1 & -\frac{n-1}{\alpha+2} \\ -\frac{n-1}{\alpha+2} & \frac{(n(\alpha+2)-(\alpha+1))(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)} \end{pmatrix}\right] \\ &= K_{n-1}(0,0)M, \end{aligned}$$

where

$$M = \begin{pmatrix} G & H \\ J & K \end{pmatrix},$$

$$\begin{aligned} G &= \frac{1}{K_{n-1}(0,0)} + \left(M_0 + \frac{\lambda}{\alpha+2}\right) - \frac{n\lambda}{\alpha+2} \\ H &= \frac{\lambda n^2}{(\alpha+1)(\alpha+3)} - \left(\frac{M_0}{\alpha+2} + \frac{(2\alpha+3)\lambda}{(\alpha+1)(\alpha+2)(\alpha+3)}\right)n + \\ &\quad \left(\frac{M_0}{\alpha+2} + \frac{\lambda}{(\alpha+2)(\alpha+3)}\right) \\ J &= -\frac{M_1}{\alpha+2}n + \left(\lambda + \frac{M_1}{\alpha+2}\right) \\ K &= \frac{M_1 n^2}{(\alpha+1)(\alpha+3)} - \left(\frac{\lambda}{\alpha+2} + \frac{(2\alpha+3)M_1}{(\alpha+1)(\alpha+2)(\alpha+3)}\right)n + \\ &\quad \left(\frac{\lambda}{\alpha+2} + \frac{M_1}{(\alpha+2)(\alpha+3)}\right) + \frac{1}{K_{n-1}(0,0)}. \end{aligned}$$

As a consequence, from (12)

$$\widetilde{L}_n^{\alpha}(x) =$$

$$= \quad L_n^{\alpha}(x) - \left(\mathbb{L}_n^{\alpha}(0)\right)^t \left(I + A\mathbb{K}_{n-1}(0)\right)^{-1} A \begin{pmatrix} \frac{(-1)^{n-1}(\alpha+1)}{(n-1)!\Gamma(\alpha+2)} & 0 \\ \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)} & \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)} \end{pmatrix} \begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix}$$

$$= \quad L_n^{\alpha}(x) - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)K_{n-1}(0,0)} \begin{pmatrix} (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \\ (-1)^{n-1} \frac{n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \end{pmatrix}^t \begin{pmatrix} G & H \\ J & K \end{pmatrix}^{-1} A \times$$

$$\begin{pmatrix} -\frac{\alpha+1}{n-1} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix},$$

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4 Proposition. For every $n \in \mathbb{N}$,

$$\widetilde{L}_n^{\alpha}(x) = \tag{14}$$

$$= L_n^{\alpha}(x) + \begin{pmatrix} -1 \\ \frac{n}{\alpha+1} \end{pmatrix}^t \begin{pmatrix} G & H \\ J & K \end{pmatrix}^{-1} A \begin{pmatrix} -(\alpha+1) & 0 \\ n-1 & n-1 \end{pmatrix} \begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix}.$$

Therefore, from (14), after some computations we get

$$\widetilde{L}_n^{\alpha}(x) = \tag{15}$$

$$= L_n^{\alpha}(x) + \frac{1}{|M|}\begin{pmatrix} \widetilde{A}_n n^2 + B_n n + C_n & \widetilde{A}'_n n^2 + B'_n n + C'_n \end{pmatrix}\begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix},$$

with

$$\begin{aligned}
\widetilde{A}_n &= \frac{2\,|A|}{(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{M_1}{(\alpha+1)K_{n-1}(0,0)} \\
B_n &= \frac{2\alpha\,|A|}{(\alpha+1)(\alpha+2)(\alpha+3)} - \frac{2\lambda}{K_{n-1}(0,0)} - \frac{M_1}{(\alpha+1)K_{n-1}(0,0)} \\
\widetilde{A}'_n &= \frac{|A|}{(\alpha+1)(\alpha+2)} + \frac{M_1}{(\alpha+1)K_{n-1}(0,0)} \\
B'_n &= \frac{\alpha\,|A|}{(\alpha+1)(\alpha+2)} - \frac{\lambda}{K_{n-1}(0,0)} - \frac{M_1}{(\alpha+1)K_{n-1}(0,0)},
\end{aligned}$$

and C_n and C'_n depend of M_0, M_1, λ and α.

Assuming that $|A| > 0$, we get

$$|M| \sim \frac{|A|}{(\alpha+1)(\alpha+2)^2(\alpha+3)} n^2.$$

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$$\frac{\widehat{\widetilde{L}}_n^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)} \sim 1 + \frac{(\alpha+1)(\alpha+2)^2(\alpha+3)}{|A|} \begin{pmatrix} \frac{\widetilde{A}_n}{n} + \frac{B_n}{n^2} + \frac{C_n}{n^3} & \frac{\widetilde{A}'_n}{n} + \frac{B'_n}{n^2} + \frac{C'_n}{n^3} \end{pmatrix} \begin{pmatrix} -\frac{\widehat{L}_{n-1}^{\alpha+1}(x)}{\widehat{L}_n^{\alpha}(x)} \\ \frac{1}{n-1} \frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{\widehat{L}_n^{\alpha}(x)} \end{pmatrix},$$

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On the other hand, if $|A| = 0$, $M_1 > 0$,

$$\frac{\widehat{\widetilde{L}}_n^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)} \sim 1 + \frac{(\alpha+1)(\alpha+3)K_{n-1}(0,0)}{M_1} \begin{pmatrix} \frac{\widetilde{A}_n}{n} + \frac{B_n}{n^2} + \frac{C_n}{n^3} & \frac{\widetilde{A}'_n}{n} + \frac{B'_n}{n^2} + \frac{C'_n}{n^3} \end{pmatrix} \begin{pmatrix} -\frac{\widehat{L}_{n-1}^{\alpha+1}(x)}{\widehat{L}_n^{\alpha}(x)} \\ \frac{1}{n-1}\frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{\widehat{L}_n^{\alpha}(x)} \end{pmatrix}.$$

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5 Theorem. For every $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{\widetilde{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} = 1 \tag{17}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

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6 Theorem. Let $\left\{\widehat{\widetilde{L}}_n^{\alpha}\right\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to (11) and $|A| > 0$. Then

$$\lim_{n\to\infty} \frac{\widehat{\widetilde{L}}_n^{\alpha}(x/n)}{n^{\alpha}} = x^{-\alpha/2} J_{\alpha+4}(2\sqrt{x}), \tag{18}$$

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7 Theorem. Let $\left\{\widetilde{\widetilde{L}}_n^{\alpha}\right\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to (11) and $|A| = 0, M_1 > 0$. Then

$$\lim_{n\to\infty} \frac{\widetilde{\widetilde{L}}_n^{\alpha}(x/n)}{n^{\alpha}} = x^{-\alpha/2} \left(J_{\alpha}(2\sqrt{x}) - \frac{\alpha+3}{\sqrt{x}} J_{\alpha+1}\left(2\sqrt{x}\right) + \frac{\alpha+3}{x} J_{\alpha+2}\left(2\sqrt{x}\right) \right),$$

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Using (11) we get

$$\left\| \widetilde{L}_n^{\alpha} \right\|_S^2 = \|L_n^{\alpha}\|_{\alpha}^2 + \mathbb{L}^{\alpha}(0)^t (I + A\mathbb{K}_{n-1}(0,0))^{-1} A\mathbb{L}^{\alpha}(0).$$

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Taking into account

$$\begin{vmatrix} 0 & u^t \\ v & B \end{vmatrix} = -|B|\, u^t B^{-1} v$$

where

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \; v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

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$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

thus

$$\begin{aligned}
\left\|\widetilde{L}_n^\alpha\right\|_S^2 &= \|L_n^\alpha\|_\alpha^2 - \frac{1}{|I + A\mathbb{K}_{n-1}(0,0)|}\begin{vmatrix} 0 & \mathbb{L}_n^\alpha(0)^t \\ A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix} \\
&= \frac{\|L_n^\alpha\|_\alpha^2}{|I + A\mathbb{K}_{n-1}(0,0)|}\left(|I + A\mathbb{K}_{n-1}(0,0)| + \begin{vmatrix} 0 & \mathbb{L}_n^\alpha(0)^t/\,\|L_n^\alpha\|_\alpha^2 \\ -A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix}\right) \\
&= \frac{\|L_n^\alpha\|_\alpha^2}{|I + A\mathbb{K}_{n-1}(0,0)|}\begin{vmatrix} 1 & \mathbb{L}_n^\alpha(0)^t/\,\|L_n^\alpha\|_\alpha^2 \\ -A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix}.
\end{aligned}$$

Using (11) we get

$$\left\|\widetilde{L}_n^\alpha\right\|_S^2 = \|L_n^\alpha\|_\alpha^2 + \mathbb{L}^\alpha(0)^t (I + A\mathbb{K}_{n-1}(0,0))^{-1} A\mathbb{L}^\alpha(0).$$

Taking into account

$$\begin{vmatrix} 0 & u^t \\ v & B \end{vmatrix} = -|B|\, u^t B^{-1} v$$

where

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

thus

$$\begin{aligned}
\left\|\widetilde{L}_n^\alpha\right\|_S^2 &= \|L_n^\alpha\|_\alpha^2 - \frac{1}{|I + A\mathbb{K}_{n-1}(0,0)|} \begin{vmatrix} 0 & \mathbb{L}_n^\alpha(0)^t \\ A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix} \\
&= \frac{\|L_n^\alpha\|_\alpha^2}{|I + A\mathbb{K}_{n-1}(0,0)|} \left(|I + A\mathbb{K}_{n-1}(0,0)| + \begin{vmatrix} 0 & \mathbb{L}_n^\alpha(0)^t / \|L_n^\alpha\|_\alpha^2 \\ -A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix} \right) \\
&= \frac{\|L_n^\alpha\|_\alpha^2}{|I + A\mathbb{K}_{n-1}(0,0)|} \begin{vmatrix} 1 & \mathbb{L}_n^\alpha(0)^t / \|L_n^\alpha\|_\alpha^2 \\ -A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix}.
\end{aligned}$$

Finally, using the fact that

$$I + A\mathbb{K}_n(0,0) = I + A\mathbb{K}_{n-1}(0,0) + \frac{A}{\|L_n^\alpha\|_\alpha^2} \mathbb{L}_n^\alpha(0)\mathbb{L}_n^\alpha(0)^t,$$

then

$$\frac{\left\| \widetilde{L}_n^\alpha \right\|_S^2}{\left\| L_n^\alpha \right\|_\alpha^2} = \frac{|I + A\mathbb{K}_n(0,0)|}{|I + A\mathbb{K}_{n-1}(0,0)|}. \tag{19}$$

then

$$\frac{\left\|\widetilde{L}_n^{\alpha}\right\|_S^2}{\|L_n^{\alpha}\|_{\alpha}^2} = \frac{|I + A\mathbb{K}_n(0,0)|}{|I + A\mathbb{K}_{n-1}(0,0)|}. \tag{19}$$

Therefore

then

$$\frac{\left\|\widetilde{L}_n^{\alpha}\right\|_S^2}{\left\|L_n^{\alpha}\right\|_{\alpha}^2} = \frac{|I + A\mathbb{K}_n(0,0)|}{|I + A\mathbb{K}_{n-1}(0,0)|}. \tag{19}$$

Therefore

8 Proposition. Let $\left\{\widehat{\widetilde{L}}_n^{\alpha}\right\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to (11). Then

$$\lim_{n\to\infty} \frac{\left\|\widetilde{L}_n^{\alpha}\right\|_S^2}{\left\|L_n^{\alpha}\right\|_{\alpha}^2} = 1.$$

4. The connection formula

9 Theorem. For every $n \in \mathbb{N}$

$$\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha+2}(x) + A_{n,\alpha} L_{n-1}^{\alpha+2}(x) + B_{n,\alpha} L_{n-2}^{\alpha+2}(x) \tag{20}$$

where

$$\begin{aligned} A_{n,\alpha} &= 2n - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)} \left(\widetilde{\mathbb{L}}_n^{\alpha}(0)\right)^t A \begin{pmatrix} -\frac{1}{n-1} \\ \frac{1}{(\alpha+1)} \end{pmatrix} \\ &\sim 2n - (\alpha+1)(\alpha+2) \\ B_{n,\alpha} &= n(n-1) - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)} \left(\widetilde{\mathbb{L}}_n^{\alpha}(0)\right)^t A \begin{pmatrix} -1 \\ \frac{n}{\alpha+1} \end{pmatrix} \\ &\sim n(n-1) - (\alpha+1)(\alpha+2)(n-1). \end{aligned}$$

In other words, the sequence of Laguerre Sobolev Type orthogonal polynomials $\left\{\widetilde{L}_n^{\alpha}\right\}_{n\geq 0}$ is quasi-orthogonal of order 2, with respect to $\{L_n^{\alpha}\}_{n\geq 0}$.

5. The five term recurrence formula

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10 Proposition. If p and q are polynomials with real coefficients, then

$$\left\langle x^2 p, q \right\rangle_S = \left\langle p, x^2 q \right\rangle_S . \tag{21}$$

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11 Theorem. (The five term recurrence formula) For every $n \in \mathbb{N}$

$$x^2 \widetilde{L}_n^\alpha(x) = \widetilde{L}_{n+2}^\alpha(x) + a_{n,n+1}\widetilde{L}_{n+1}^\alpha(x) + a_{n,n}\widetilde{L}_n^\alpha(x) + a_{n,n-1}\widetilde{L}_{n-1}^\alpha(x) + a_{n,n-2}\widetilde{L}_{n-2}^\alpha(x). \tag{22}$$

where$\widetilde{L}_{-1}^\alpha(x) = \widetilde{L}_{-2}^\alpha(x) = 0$ and

$$\begin{aligned} a_{n,n+1} &\sim 4n \\ a_{n,n} &\sim 6n^2 \\ a_{n,n-1} &\sim 4n^3 \\ a_{n,n-2} &\sim n^4. \end{aligned}$$

5.1. Example

If p, q are polynomials with real coefficients, we introduce the inner product

$$\langle p,q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + Mp(0)q(0) + 4Np'(0)q'(0). \tag{23}$$

Let $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ be the sequence of monic Laguerre-Sobolev polynomials orthogonal with respect to (23) and $\{P_n\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the following inner product

$$\langle p,q\rangle_H = \int_{-\infty}^\infty p(x)q(x)\left|x\right|^{2\alpha} e^{-x^2}dx + Mp(0)q(0) + Np''(0)q''(0), \tag{24}$$

$M, N \in \mathbb{R}^+$. Then

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$M, N \in \mathbb{R}^+$. Then

12 Proposition. For every $n \in \mathbb{N}$

$$\begin{aligned} P_{2n}(x) &= \widetilde{L}_n^{\alpha-1/2}(x^2) \\ P_{2n+1}(x) &= xL_n^{\alpha+1/2}(x^2). \end{aligned}$$

6. The holonomic equation

Let

$$\begin{aligned} f(x;n) &= A_{n,\alpha} + (x - (2n + 1 + \alpha)) \\ M_n &= B_{n,\alpha} - (n-1)(n + \alpha + 1) \\ K_n &= 1 - \frac{B_{n-1,\alpha}}{(n-2)(n+\alpha)} \\ g(x;n) &= A_{n-1,\alpha} + \frac{B_{n-1,\alpha}(x - (2n + \alpha - 1))}{(n-2)(n+\alpha)}. \end{aligned}$$

$$\begin{aligned} \phi(x;n) &= nx + A_{n,\alpha}n - n - B_{n,\alpha} - A_{n,\alpha} - n^2 \\ \sigma(x;n) &= B_{n,\alpha}x + A_{n,\alpha}\left(-1 - \alpha + n\alpha + n^2\right) - B_{n,\alpha}(1 + n + \alpha) + n(\alpha - n^2 - n\alpha + 1) \end{aligned}$$

then

13 Proposition. Let $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ be the sequence of monic polynomials orthogonal with respect to (11). Then, the differential operators $\mathcal{J}_n$ and $\mathcal{K}_n$ defined by

$$\begin{aligned}\mathcal{J}_n &= F(x;n)D + G(x;n)I\\ \mathcal{K}_n &= F(x;n)D + J(x;n)I\end{aligned}$$

where D is the derivate operator, I the identity operator, and

$$\begin{aligned}F(x;n) &= x\left[g(x;n)f(x;n) - K_n M_n\right],\\ G(x;n) &= \phi(x;n)\left(K_n - g(x;n)\right),\\ H(x;n) &= f(x;n)\sigma(x;n) - M_n\phi(x;n),\\ J(x;n) &= M_n\tau(x;n) - \upsilon(x;n)f(x;n),\\ K(x;n) &= \tau(x;n)g(x;n) - K_n\upsilon(x;n),\end{aligned}$$

satisfy

$$\begin{aligned}\mathcal{J}_n\left(\widetilde{L}_n^\alpha\right) &= H(x;n)\widetilde{L}_{n-1}^\alpha, \text{ (the lowering operator)}.\\ \mathcal{K}_n\left(\widetilde{L}_{n-1}^\alpha\right) &= K(x;n)\widetilde{L}_n^\alpha, \text{ (the raising operator)}.\end{aligned}$$

14 Theorem. Let $\left\{\widetilde{L}_n^a\right\}_{n\geq 0}$ be the sequence of monic Laguerre-Sobolev polynomials orthogonal with respect to (11). Then

$$A(x;n)\left(\widetilde{L}_n^{\alpha}(x)\right)'' + B(x;n)\left(\widetilde{L}_n^{\alpha}(x)\right)' + C(x;n)\widetilde{L}_n^{\alpha}(x) = 0,$$

where

$$\begin{aligned}
A(x;n) &= F^2(x;n) \\
B(x;n) &= F(x;n)\left[F'(x;n) + G(x;n) + J(x;n) - \frac{F(x;n)H'(x;n)}{H(x;n)}\right] \\
C(x;n) &= F(x;n)G'(x;n) + J(x;n)G(x;n) - \frac{F(x;n)G(x;n)H'(x;n)}{H(x;n)} - K(x;n)H(x;n).
\end{aligned}$$

Thank you for your attention