Edge-adaptive multiresolution with normal offsets

Maarten Jansen

Joint with

Ward Van Aerschot & Adhemar Bultheel

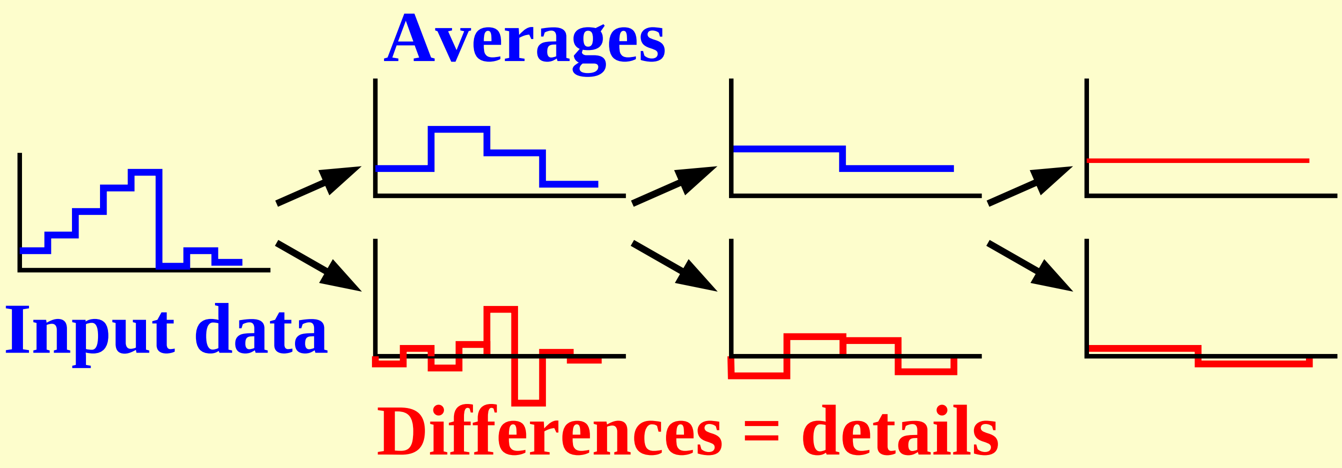
K.U.Leuven

Rolling Waves in Leuven 2008

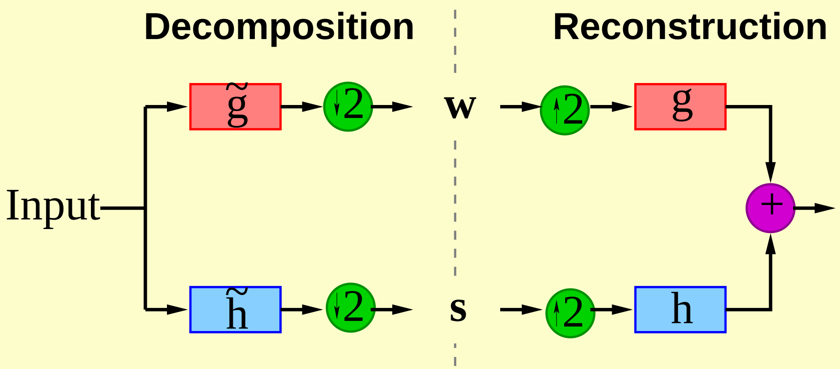
16 December 2008

1. Life jacket for a sea of wavelets

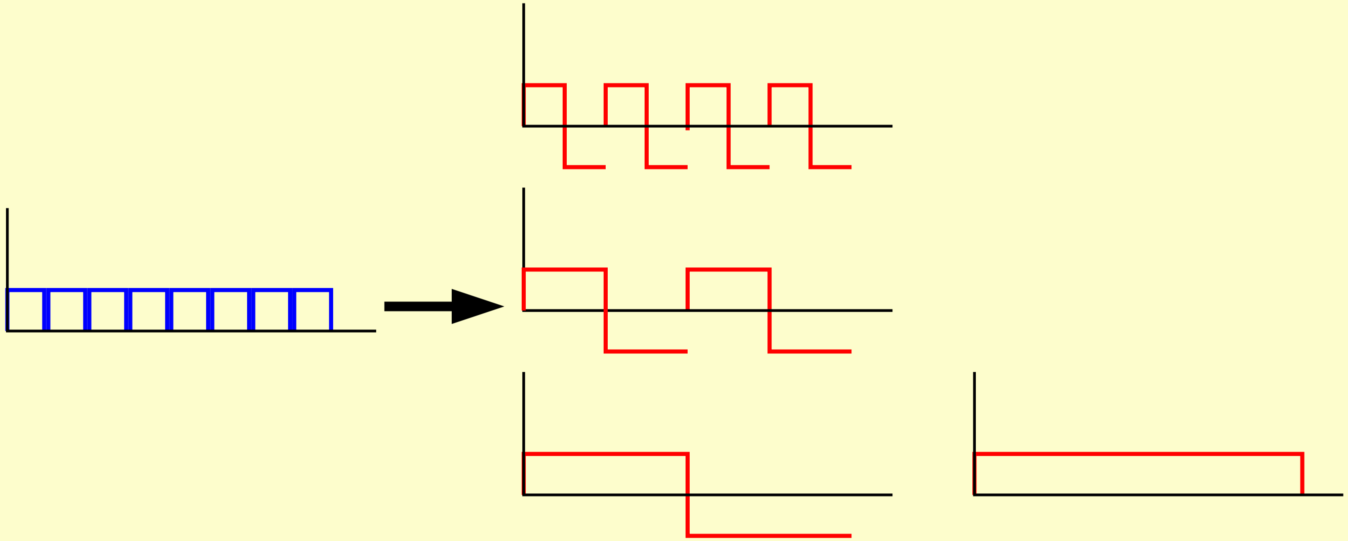
Actually, you should (have) learn(t) to swim, for instance, using Adhemar's introductory notes or his course text



This is a repeated (multiscale) **filterbank**

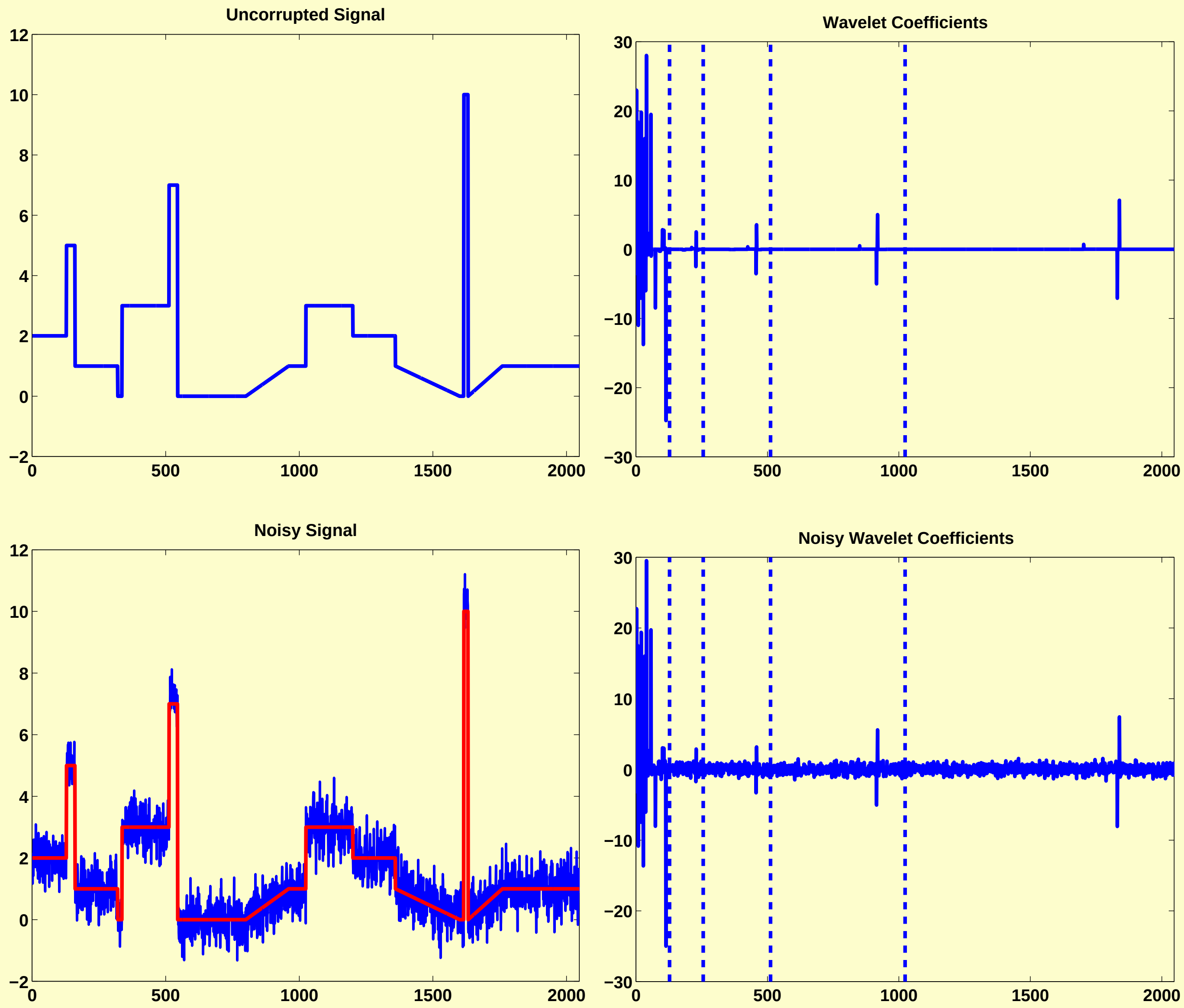


Basis functions



Local in time/space & frequency/scale

Example



Uncorrupted Signal
Wavelet Coefficients
Noisy Signal
Noisy Wavelet Coefficients

Basic concepts

Basic idea: locality of basis functions (time/space **&** frequency)

$\longrightarrow \left\{ \begin{array}{l} \bullet \quad \text{Sparsity} \\ \bullet \quad \text{Multiresolution} \end{array} \right.$

$\longrightarrow$ {

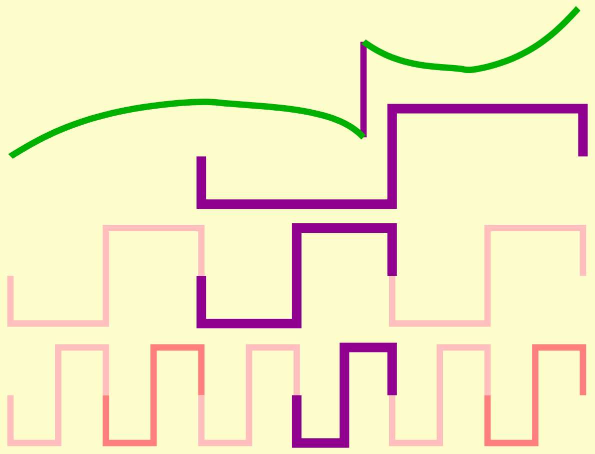
- **Nonlinear** processing for **piecewise** smooth data

  Thresholds: select **'largest'** $=$ nonlinear

  $\leftrightarrow$ **'first'** $=$ linear
- Scale dependent/ across scale processing

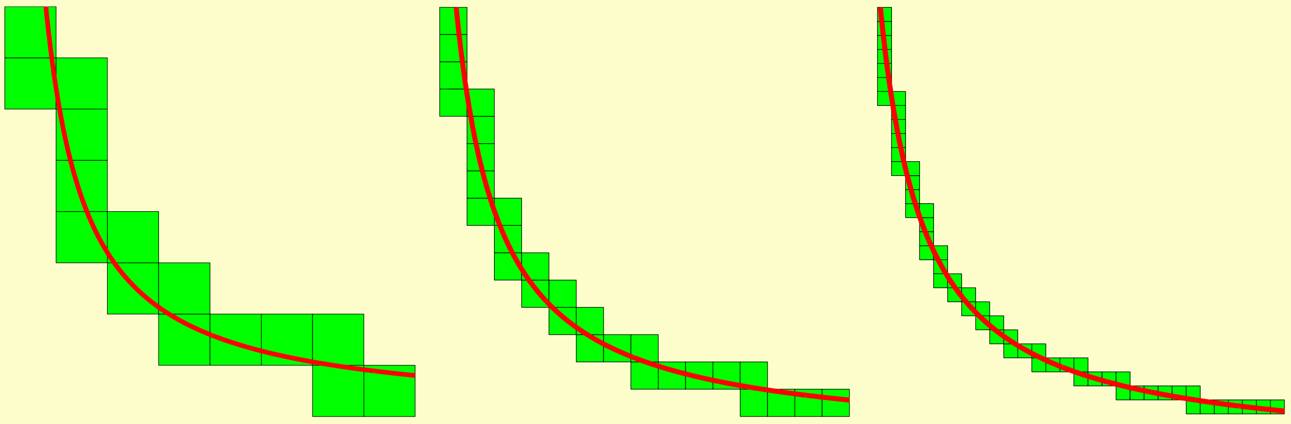
2D $>$ (1D)^2

1D: point singularities



- # basis fun's with overlap singularity: constant / scale
- $\longrightarrow \mathcal{O}(\log(\#\text{data}))$
- Nonlinear error singularity $\leq$ linear error regular part

2D: line singularities



Wavelets too "passive" $\rightarrow$ **active** search for edges

2. The building blocks: the lifting scheme

Three basic steps: **Split** — **Predict (Dual)** — **Update (Primal)**

- **Observations**

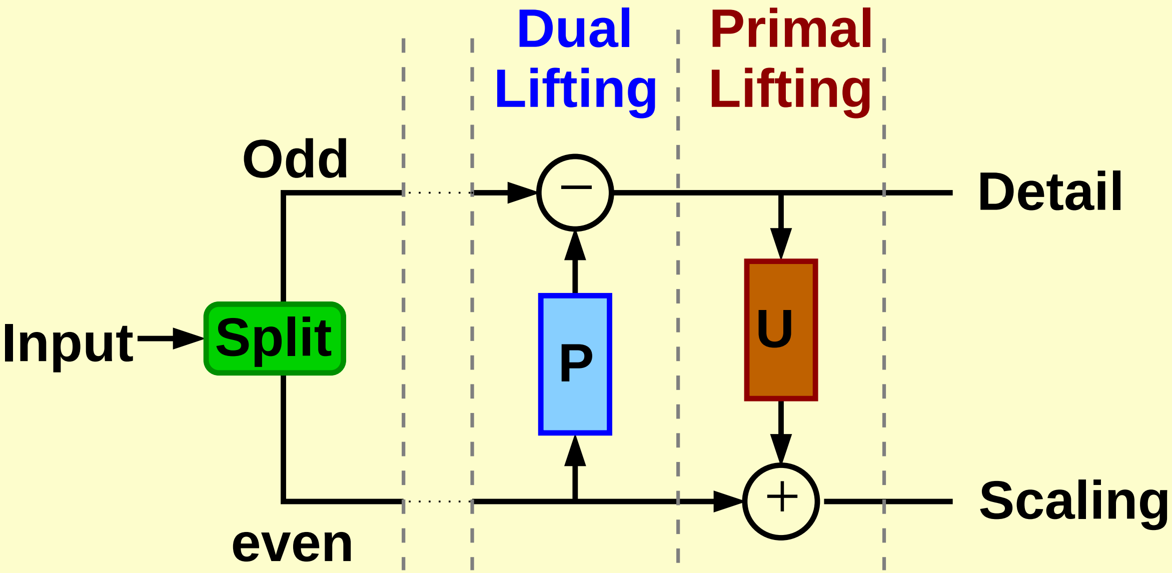
  $Y_i = f(t_i) + \varepsilon_i$

- **Initialisation**

  $$\begin{aligned} S_{J,k} &= Y_k \\ t_{J,k} &= t_k \end{aligned}$$

- **Iteration**

  $\longrightarrow$ **Multiscale**

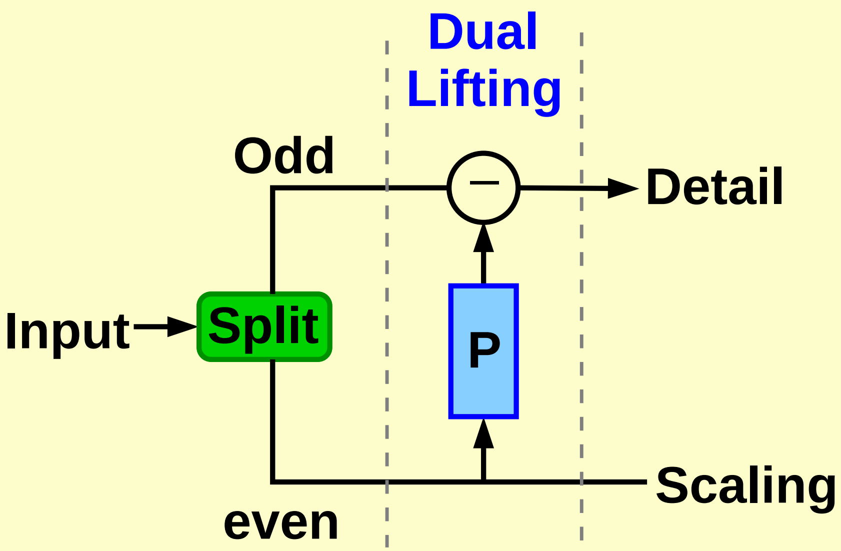


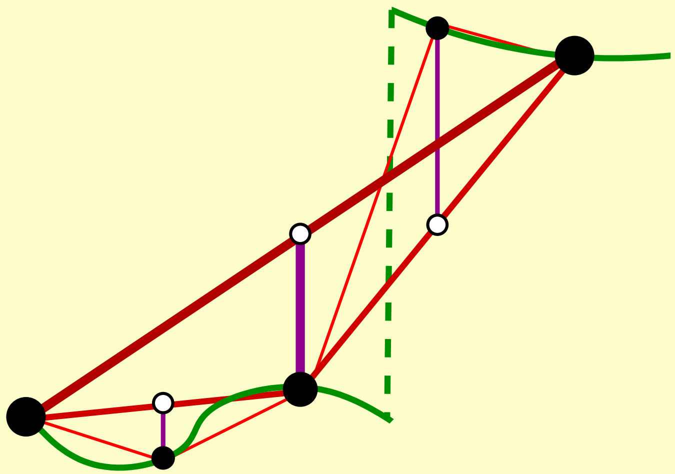
$t_{j,k} = t_{j+1,2k}$

$W_{j,k} = S_{j+1,2k+1} - \mathbf{P}_{\boldsymbol{t}_{j+1}}(S_{j+1,2k}, S_{j+1,2k+2})$

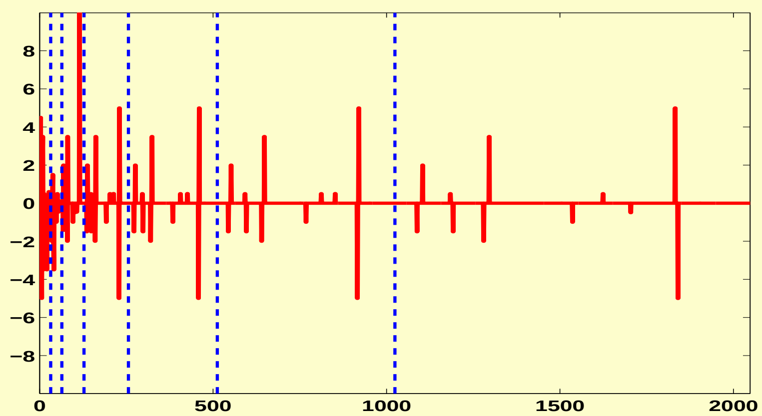
$S_{j,k} = S_{j+1,2k} + \mathbf{U}_{\boldsymbol{t}_{j+1}}(W_{j,k-1}, W_{j,k})$

An example: linear interpolating prediction, no update





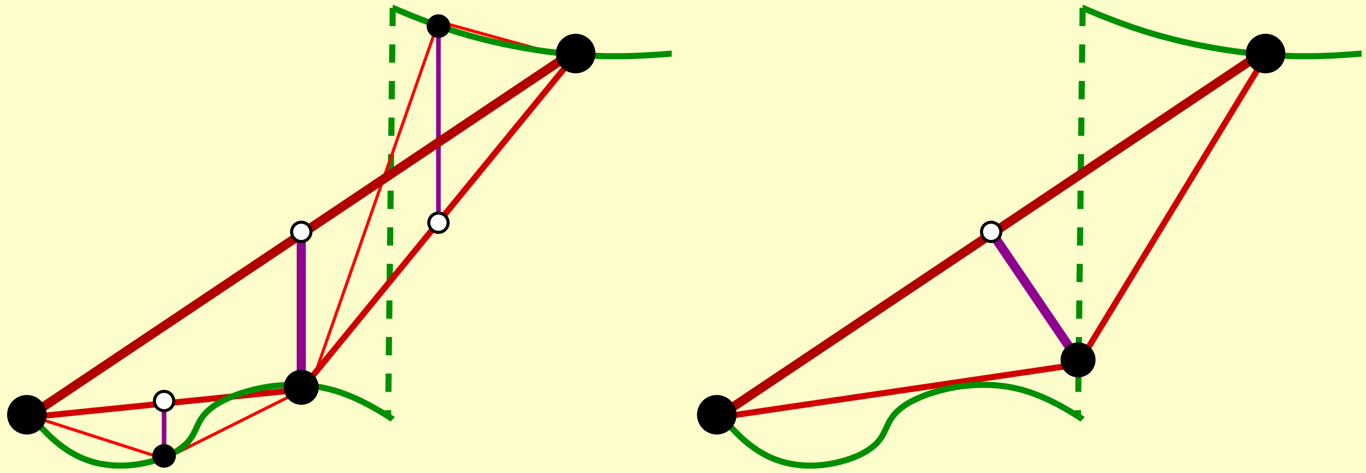
may proceed from coarse to fine



Only 115 out of 2048 coefficients are non-zero!

Normal offsets (instead of vertical)

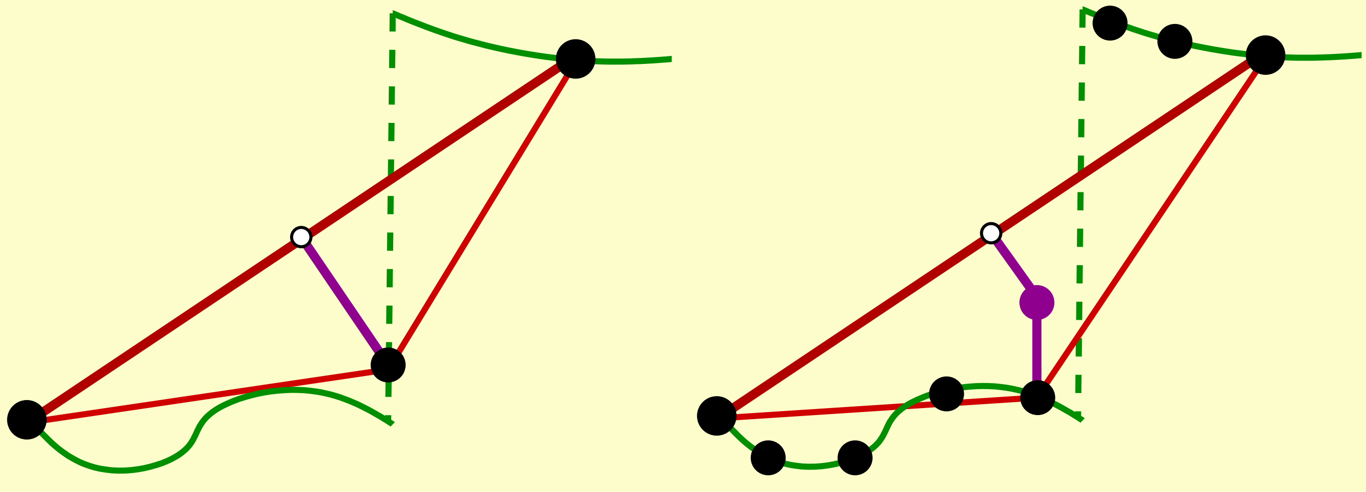
We can do better (sparser representation)!



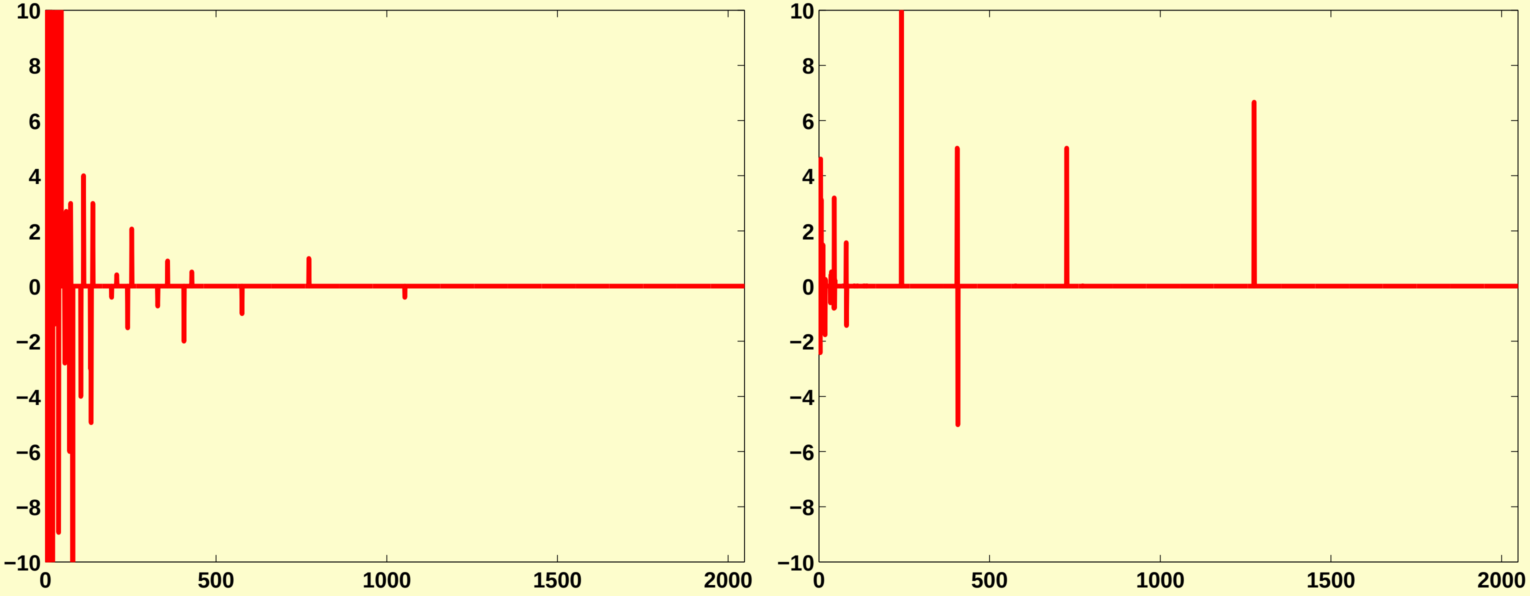
Normal Offsets:

- Coefficients also carry directional information
- Non-linear transform
- **must** proceed from coarse to fine

Discretisation: normal and local offsets



Example



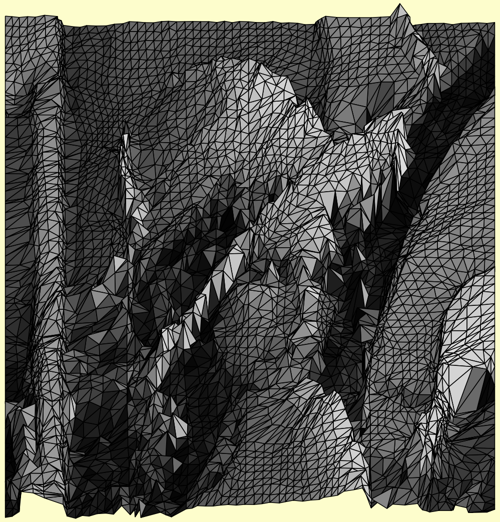
- **Local offsets** correct for discretisation effects
- **Normal and local offsets** together are sparser than vertical offsets (45 + 49 = 94 non-zeros)

The effect is substantial in 2-D

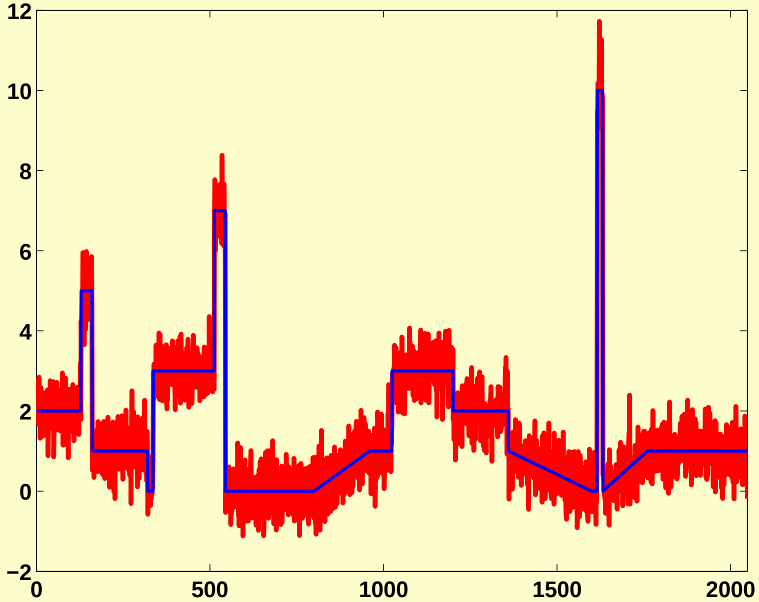
2-D Approximation error rate

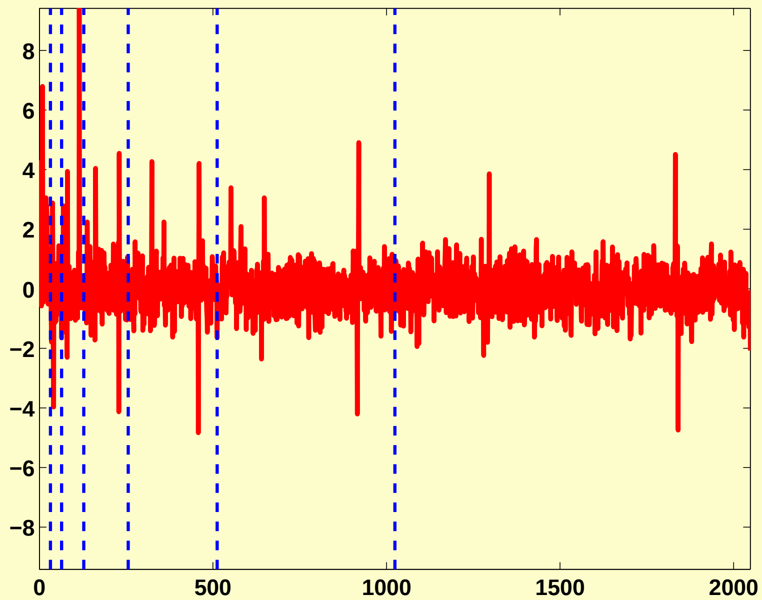
(Jansen, e.a., Appl. Comp. Harm. Anal., 2005)

1. **Fourier:** $\|f - f_n\| = \mathcal{O}(n^{-1/4})$.
2. **Wavelets:** $\|f - f_n\| = \mathcal{O}(n^{-1/2})$.
3. **Normal offsets:** $\|f - f_n\| = \mathcal{O}(n^{-1})$.
4. Similar to Curvelets, wedgelets, bandelets, contourlets, . . .



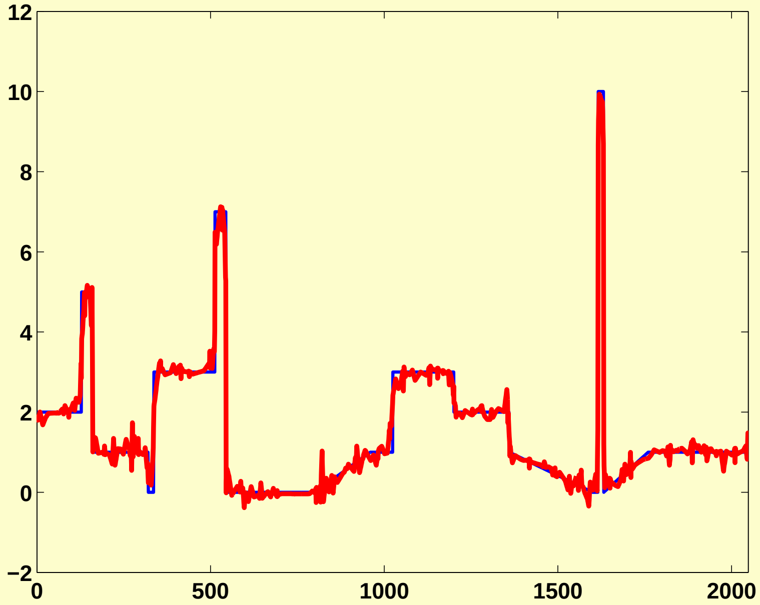
Thresholding vertical offsets



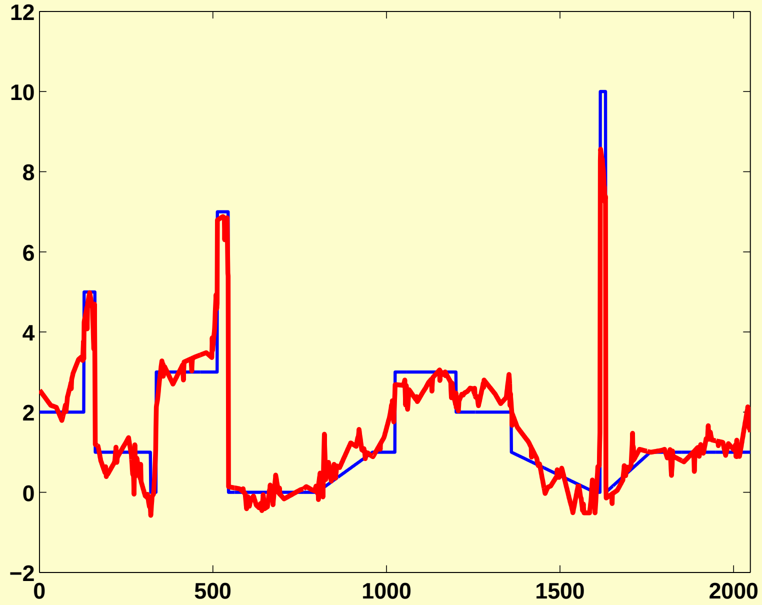


Thresholding: $\longleftarrow$ sparsity

Vertical offsets are no wavelet coeff.



Wavelet thresholding



Vertical offset thresholding

Vertical offset smoothing is biased, especially near discontinuities

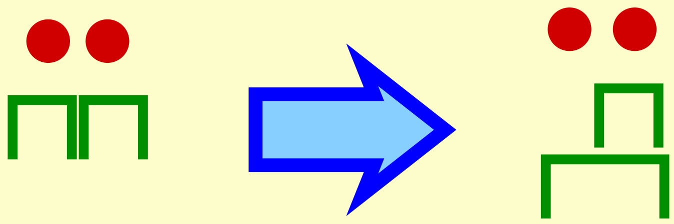
The basis functions with and without update

Vertical offsets with constant extrapolating prediction
(the simplest case)

$$\begin{aligned} W_{j,k} &= S_{j+1,2k+1} - S_{j+1,2k} \\ S_{j,k} &= S_{j+1,2k} \end{aligned}$$

Transform matrix: $\tilde{V_1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Inverse matrix has basis in its columns: $\tilde{V_1}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

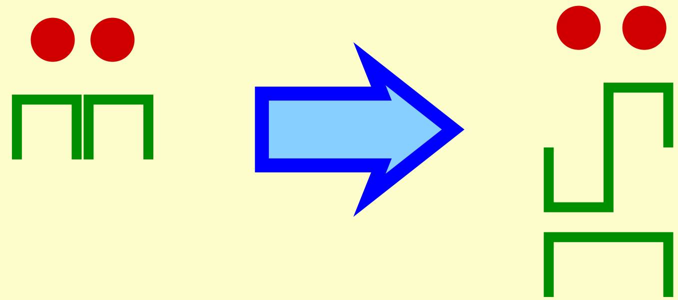


Wavelet coefficients: Prediction + Update

$$\begin{aligned} W_{j,k} &= S_{j+1,2k+1} - S_{j+1,2k} \\ S_{j,k} &= S_{j+1,2k} + \frac{1}{2} W_{j,k} \\ &= \frac{S_{j+1,2k+1} + S_{j+1,2k}}{2} \end{aligned}$$

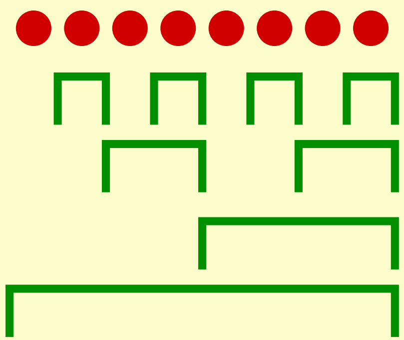
$\tilde{W_1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{bmatrix}$

$\tilde{W_1}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$



Hierarchical basis and wavelet basis

- **Hierarchical basis**

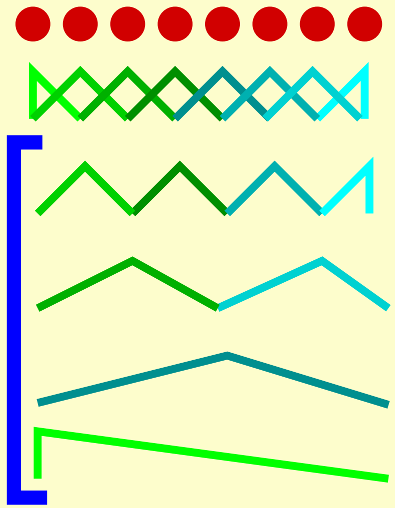


L_2-converging non-trivial decompositions of 0

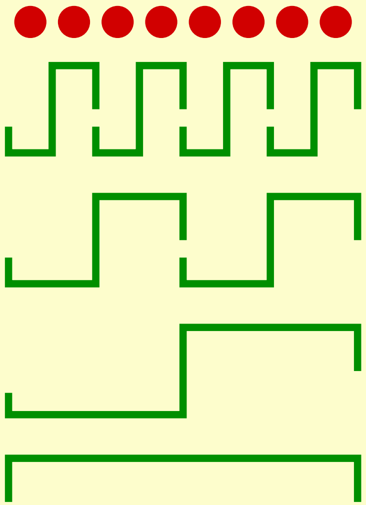
$0 = \varphi_{00} - \psi_{00} - \psi_{10} - \psi_{20} - \ldots$

Or:

$0 = \psi_{00} - \psi_{11} - \psi_{22} - \ldots$



- **Wavelet basis** (Haar-wavelets)

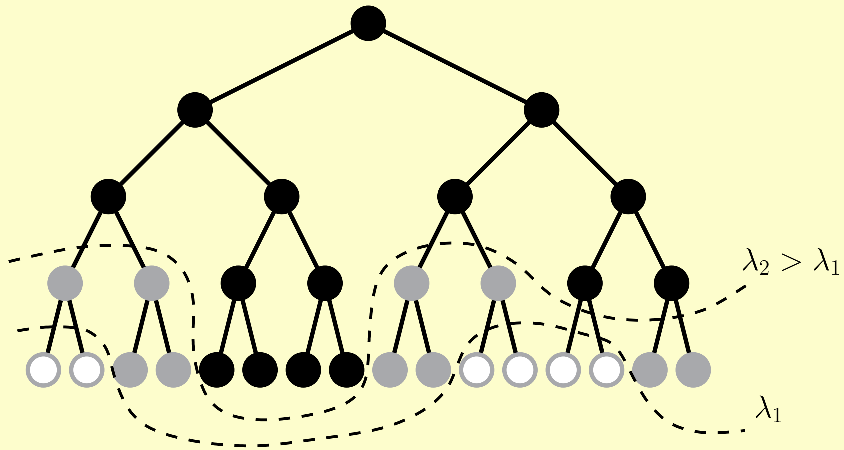


  - Hierarchical bases allow non-linear processing
  - Riesz-stability in L_2 requires $\boxed{\int_{-\infty}^{\infty} \psi_{j,k} dx = 0}$

    (Jansen & Oonincx, Springer Verlag, 2005)

3. Use in image compression

- Normal offset decomposition is quadtree-structured
- Trade-off $\lambda_{\text{tree}} = -\frac{\Delta D(\text{tree})}{\Delta R(\text{tree})}$ $D(\text{tree}) + \lambda R(\text{tree})$
  - Distortion: approximation + quantisation errors
  - Rate: (penalty term) bits for coding coeffi cients, tesselation, quantisation



Compression