ORTHOGONALITY, INTERPOLATION AND QUADRATURES ON THE UNIT CIRCLE AND THE INTERVAL $[-1, 1]$

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"Dedicated to Adhemar Bultheel on the occassion of his 60-th birthday"

Joint work with:

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Preliminary results

Starting point: Erdös-Turán Theorem.
$\sigma(x)$: weight function on $[a,b]=[-1,1]$
Q_n : n-th orthogonal polynomial
$x_{1,n},\dots,x_{n,n}$ the n distinct zeros of $Q_n(x)$
f: a function defined on $[-1,1]$
$P_{n-1}(f,x)\in\Pi_{n-1}$ s.t. $P_{n-1}(f,x_{j,n})=f(x_{j,n})$, $j=1,\dots,n$

$$\lim_{n\to\infty}\|f-P_{n-1}(f,\cdot)\|_{2,\sigma}=\left[\int_{-1}^{1}|f(x)-P_{n-1}(f,x)|^2\sigma(x)dx\right]^{1/2}=0$$

($\{x_{j,n}\}_{j=1}^{n}$; $n=1,2,\dots$ nodes in "Gauss-type interpolation")

$$I_\sigma(P_{n-1}(f,x))=\int_{-1}^{1}P_{n-1}(f,x)\sigma(x)dx=\sum_{j=1}^{n}A_{j,n}f(x_{j,n})=I_n^\sigma(f)$$

(Gauss-Christophel Quadrature formulas) $I_n^\sigma(p)=I_\sigma(p)$, $\forall p\in\Pi_{2n-1}$

Interpolatory quadrature formulas. Product integration rules

$\alpha(x)$: possibly complex function on $[-1,1]$
$I_\alpha(f) = \int_{-1}^{1} f(x)\alpha(x)dx$
$\sigma(x)$: weight function on $[-1,1]$

$$I_\alpha(P_{n-1}(f,x)) = \sum_{j=1}^{n} A_{j,n} f(x_j,n) = I_n^\alpha(f) = I_\alpha(f), \forall f \in \Pi_{n-1}$$

Assume: $\displaystyle\int_{-1}^{1} \frac{|\alpha(x)|^2}{\sigma(x)} dx < +\infty$. (Sloan-Smith, 1982)

1. $\displaystyle\lim_{n\to\infty} I_n^\alpha(f) = I_\alpha(f)$
2. $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} |A_{j,n}| f(x_{j,n}) = \int_{-1}^{1} f(x)|\alpha(x)|dx$. (Stability)

Radau-type and Lobatto-type interpolation

$\sigma(x)$ weight function on $[-1, 1]$

1. $\sigma^{\pm}(x) = (1 \pm x)\sigma(x)$
   $x_{1,n}, \dots, x_{n,n}$ the zeros of the n-th orthogonal polynomial,
   $x_{0,n} \in \{\pm 1\}$
   Set: $P_n^{\pm}(f, x) \in \Pi_n : P_n^{\pm}(f, x_{j,n}) = f(x_{j,n}),\ j = 0, 1, \dots, n$
2. $\tilde{\sigma}(x) = (1 - x^2)\sigma(x)$
   $x_{1,n}, \dots, x_{n,n}$ the zeros of the n-th orthogonal polynomial,
   $x_{0,n} = -1,\ x_{n+1,n} = 1$
   Set: $\tilde{P}_n(f, x) \in \Pi_n : \tilde{P}_n(f, x_{j,n}) = f(x_{j,n}),\ j = 0, 1, \dots, n + 1$

$I_\sigma(P_n^{\pm}(f, \cdot))$: Gauss-Radau Quadrature formula.
$I_\sigma(\tilde{P}_n(f, \cdot))$: Gauss-Lobatto Quadrature formula.

Aim of the talk

1. Investigate the L_2-convergence w.r.t. $\sigma(x)$ of the interpolant sequences $\{P_n^{\pm}(f,x)\}$ and $\{\tilde{P}_n(f,x)\}$

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1. Investigate the L_2-convergence w.r.t. $\sigma(x)$ of the interpolant sequences $\{P_n^{\pm}(f,x)\}$ and $\{\tilde{P}_n(f,x)\}$
2. Application: Product integration rules taking as nodes 1 or -1 or both.

Passing from $[-1,1]$ to $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,
Joukowsky Transformation $x = \frac{1}{2}(z + z^{-1})$

$$\begin{array}{rcl} [-1,1] & \longleftrightarrow & \mathbb{T} \\ \mathbb{C} \setminus [-1,1] & \longleftrightarrow & \mathbb{D} = \{z \in C : |z| < 1\} \end{array}$$

$\sigma(x)$ weight function on $[-1,1]$
$\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ weight function on $\mathbb{T}$

$$I_\sigma(f) = \int_{-1}^{1} f(x)\sigma(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta = \frac{1}{2} I_\omega(g)$$

$g(e^{i\theta}) = f(\cos\theta)$, $\|f\|_{2,\sigma} = \frac{1}{\sqrt{2}}\|g\|_{2,\omega}$

Interpolation on the unit circle

g : 2π-periodic function or equivalently, defined on $\mathbb{T}$
Interpolants: trigonometric polynomials or more generally "Laurent Polynomials".

$$\Lambda_{p,q} = span\langle z^j : p \leq j \leq q\rangle$$

Nodes on $\mathbb{T}$ associated with a weight function $\omega(\theta)$, $\theta \in [-\pi, \pi]$

$$\langle g, h\rangle_\omega = \int_{-\pi}^{\pi} g(e^{i\theta})\overline{h(e^{i\theta})}\omega(\theta)d\theta$$

$\{\rho_n(z)\}_0^\infty$: the sequence of monic Szegő polynomials.

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$\{\rho_n(z)\}_0^\infty$: the sequence of monic Szegő polynomials.

For each $n \geq 1$ the nodes of $\rho_n(z)$ lie in $\mathbb{D} = \{z : |z| < 1\}$

Interpolation on the unit circle

Take $\tau_n \in \mathbb{T}$ and set

$B_n(z, \tau_n) = \rho_n(z) + \tau_n \rho_n^*(z)$ (para-orthogonality)

$\left(\rho_n^*(z) = z^n \rho_{n*}(z), \ \rho_{n*}(z) = \overline{\rho_n(1/\overline{z})} \right)$

$B_n(z, \tau_n)$: has exactly n distinct zeros on $\mathbb{T}$, $\{z_{j,n}\}_{j=1}^n$

$\mathfrak{R}_n = \Lambda_{-p(n),q(n)}$; $p(n) + q(n) = n$ and

$\lim_{n\to\infty} p(n) = \lim_{n\to\infty} q(n) = \infty$, $\bigcup \mathfrak{R}_\mathfrak{n} = \Lambda$

$L_{n-1}(g, z) \in \mathfrak{R}_{n-1}$: $L_{n-1}(g, z_{j,n}) = g(z_{j,n})$, $j = 1, \ldots, n$

Interpolation on the unit circle

Setting: $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$

1. $I_\omega(L_{n-1}(g,\cdot)) = \sum_{j=1}^{n} \lambda_{j,n} g(z_{j,n}) = I_n^\omega(g) = I_\omega(g)$,
   $\forall g \in \mathfrak{R}_{n-1} \cdot \mathfrak{R}_{n-1*}$ $(\mathfrak{R}_{n-1*} = \{g / g_* \in \mathfrak{R}_{n-1}\})$
   (Szegő quadrature formula)
2. $$\lim_{n\to\infty} \|L_{n-1}(g,\cdot) - g\|_{2,\omega} = \left[\int_{-\pi}^{\pi} |g(e^{i\theta}) - L_{n-1}(g,e^{i\theta})|^2 \omega(\theta)d\theta\right]^{1/2} = 0$$

Product integration rules on $\mathbb{T}$

$$I_\beta(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\beta(\theta)d\theta$$

$\beta(\theta)$: a possibly complex function on $\mathbb{T}$.
$\omega(\theta)$: a weight function on $\mathbb{T}$ generating the interpolant $L_{n-1}(g,z) \in \mathfrak{R}_{n-1} = \Lambda_{-p(n-1),q(n-1)}$

$$I_\beta(L_{n-1}(g,\cdot)) = \sum_{j=1}^{n} \lambda_{j,n}^{\beta} g(z_{j,n}) = I_\beta(g), \ \forall g \in \mathfrak{R}_{n-1}$$

Product integration rules on $\mathbb{T}$

Assume: $\displaystyle\int_{-\pi}^{\pi} \frac{|\beta(\theta)|^2}{\omega(\theta)} d\theta < +\infty.$

1. $\displaystyle\lim_{n\to\infty} I_n^{\beta}(g) = I_{\beta}(g)$
2. $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} |\lambda_{j,n}^{\beta}| g(z_{j,n}) = \int_{-\pi}^{\pi} g(e^{i\theta}) |\beta(\theta)| d\theta$

A. Bultheel, P. González-Vera, E. Hendriksen, O. Njastad.
(Analysis, I, II (1998), III (2000))
Rational functions with prescribed nodes on $\hat{\mathbb{C}}$ but not on $\mathbb{T}$,
$\alpha_k, \frac{1}{\overline{\alpha}_k}, \alpha_k \in \mathbb{D}$.
$\alpha_k = 0,\ k = 1, 2, \ldots$ poles collapse at $\{0, \infty\}$
Rational functions $\Rightarrow$ Laurent polynomials

Interpolation and L_2-convergence

f a function on $[-1,1]$, $\{x_k\}_1^n \subset (-1,1)$: $x_j \neq x_k$ if $j \neq k$.
Set $x = cos(\theta)$, $\theta \in [0,\pi], g(\theta) = g(e^{i\theta}) = f(cos(\theta))$ and
$x_j = cos(\theta_j), \theta_j \in [0,\pi)$
$z_j = e^{i\theta_j}$, $z_{n+j} = \overline{z}_j$, $j = 1,\dots,n \Rightarrow 2n$ distinct nodes on $\mathbb{T}$.
Set:

- $L_{n-1}(g,z) \in \Lambda_{-(n-1),n}$: $L_{n-1}(g,z_j) = g(z_j)$, $j = 1,\dots,2n$
- $\hat{L}_{n-1}(g,z) \in \Lambda_{-n,n-1}$: $\hat{L}_{n-1}(g,z_j) = g(z_j)$, $j = 1,\dots,2n$
- $P_{n-1}(f,x) \in \Pi_{n-1}$: $P_{n-1}(f,x_j) = f(x_j)$, $j = 1,\dots,n$

Then:
$P_{n-1}(f,x) = L_{n-1}(g,z) = \hat{L}_{n-1}(g,z)$, $x = \cos\theta = \frac{1}{2}(z + \frac{1}{z})$, $z = e^{i\theta}$
$\Rightarrow \|f - P_{n-1}(f,\cdot)\|_{2,\sigma} = \frac{1}{\sqrt{2}}\|g - L_{n-1}(g,\cdot)\|_{2,\omega} = \frac{1}{\sqrt{2}}\|g - \hat{L}_{n-1}(g,\cdot)\|_{2,\omega}$
$\sigma(x)$: weight function on $[-1,1]$,
$\omega(\theta) = \sigma(\cos\theta))|\sin\theta|$

Interpolation and L_2-convergence

Assume $x_0, x_1, \ldots, x_n$, $n+1$ distinct nodes on $[-1,1]$
$x_0 \in \{\pm 1\}$ and $\{x_k\}_1^n \subset (-1,1)$
$P_n^{\pm}(f,x) \in \Pi_n : P_n^{\pm}(f,x_j) = f(x_j)$, $j = 0,1,\ldots,n$
(P_n^{+} means we have taken $x_0 = 1$ and P_n^{-}, $x_0 = -1$)
We have $2n+1$ nodes on $\mathbb{T}$ $\{z_j\}_{j=0}^{2n}$ $z_0 \in \{\pm 1\}$
$z_j = e^{i\theta_j}, z_{n+j} = \overline{z}_j, x_j = \cos\theta_j, j = 1,\ldots,n$, $g(e^{i\theta}) = f(\cos\theta)$
$L_n^{\pm}(g,z) \in \Lambda_{-n,n}$ such that $L_n^{\pm}(g,z_j) = g(z_j)$, $j = 0,1,\ldots,2n$
Then: $P_n^{\pm}(f,x) = L_n^{\pm}(g,z)$, $x = \frac{1}{2}(z + \frac{1}{z})$ $z = e^{i\theta}$
$\Rightarrow \|f - P_n^{\pm}(f,\cdot)\|_{2,\sigma} = \frac{1}{\sqrt{2}}\|g - L_n^{\pm}(g,\cdot)\|_{2,\omega}$
$\sigma(x)$ on $[-1,1]$, $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$,

Interpolation and L_2-convergence

Assume $x_0, x_1, \ldots, x_n, x_{n+1}$, $n+2$ distinct nodes on $[-1,1]$, with $x_0 = -1$, $x_{n+1} = 1$ and $\{x_k\}_1^n \subset (-1,1)$
$P_{n+1}(f,x) \in \Pi_{n+1} : \; P_{n+1}(f,x_j) = f(x_j)$, $j = 0,1,\ldots,n+1$
$z_0 = -1, z_{2n+1} = 1, z_j = e^{i\theta_j}, z_{n+j} = \overline{z}_j, x_j = \cos\theta_j, j = 1,\ldots,n,$
$2n+2$ distinct nodes on $\mathbb{T}$.
$L_{n+1}(g,z) \in \Lambda_{-n,n+1}$: $L_{n+1}(g,z_j) = g(z_j)$, $j = 0,1,\ldots,2n+1$
$P_{n+1}(f,x) = P_{n+1}(f,\cos\theta) = \frac{1}{2}\left[L_{n+1}(g,z) + L_{n+1}(g,\frac{1}{z})\right]$,
$z = e^{i\theta} \Rightarrow \|f - P_{n+1}(f,\cdot)\|_{2,\sigma} \leq \frac{1}{\sqrt{2}}\|g - L_{n+1}(g,\cdot)\|_{2,\omega}$
$\sigma(x)$, $x \in [-1,1]$, $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$, $\theta \in [-\pi,\pi]$

A connection between orthogonality on $[-1,1]$ and para-orthogonality on $\mathbb{T}$

$\sigma(x), x \in [-1,1]$, weight function, $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|, \theta \in [-\pi,\pi]$
$B_n(z,\tau) = \rho_n(z) + \tau\rho_n^*(z)$, $(|\tau| = 1)$

$Q_n^{(0)}(x)$: n-th orthogonal polynomial w.r.t $\sigma(x)$. Then,
$x_j = \cos\theta_j$, $j = 1,\ldots,n$ are the zeros of $Q_n^{(0)}(x)$ iff
$z_j = e^{i\theta_j}$, $z_{n+j} = \overline{z}_j$, $j = 1,\ldots,n$ are the zeros of

$$B_{2n}(z,1) = \rho_{2n}(z) + \rho_{2n}^*(z)$$

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$$B_{2n}(z,1) = \rho_{2n}(z) + \rho_{2n}^*(z)$$

$Q_n^{(1)}(x)$: n-th orthogonal polynomial w.r.t $(1-x)\sigma(x)$.Then,
$x_j = \cos\theta_j$, $j = 1,\dots,n$ are the zeros of $Q_n^{(1)}(x)$ iff
$z_j = e^{i\theta_j}$, $z_{n+j} = \overline{z}_j$, $j = 1,\dots,n$ and $z_0 = 1$ are the zeros of

$$B_{2n+1}(z,-1) = \rho_{2n+1}(z) - \rho_{2n+1}^*(z)$$

A connection between orthogonality on $[-1,1]$ and para-orthogonality on $\mathbb{T}$

$\sigma(x), x \in [-1,1]$, weight function, $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|, \theta \in [-\pi,\pi]$
$B_n(z,\tau) = \rho_n(z) + \tau\rho_n^*(z)$, $(|\tau| = 1)$

$Q_n^{(2)}(x)$: n-th orthogonal polynomial w.r.t $(1+x)\sigma(x)$. Then,
$x_j = \cos\theta_j$, $j = 1,\ldots,n$ are the zeros of $Q_n^{(2)}(x)$ iff
$z_j = e^{i\theta_j}$, $z_{n+j} = \overline{z}_j$, $j = 1,\ldots,n$ and $z_0 = -1$ are the zeros of

$$B_{2n+1}(z,1) = \rho_{2n+1}(z) + \rho_{2n+1}^*(z)$$

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$\sigma(x), x \in [-1,1]$, weight function, $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|, \theta \in [-\pi,\pi]$
$B_n(z,\tau) = \rho_n(z) + \tau\rho_n^*(z)$, $(|\tau| = 1)$

$Q_n^{(2)}(x)$: n-th orthogonal polynomial w.r.t $(1+x)\sigma(x)$. Then, $x_j = \cos\theta_j$, $j = 1,\ldots,n$ are the zeros of $Q_n^{(2)}(x)$ iff $z_j = e^{i\theta_j}$, $z_{n+j} = \overline{z}_j$, $j = 1,\ldots,n$ and $z_0 = -1$ are the zeros of

$$B_{2n+1}(z,1) = \rho_{2n+1}(z) + \rho_{2n+1}^*(z)$$

$Q_n^{(3)}(x)$: n-th orthogonal polynomial w.r.t $(1-x^2)\sigma(x)$.Then, $x_j = \cos\theta_j$, $j = 1,\ldots,n$ are the zeros of $Q_n^{(3)}(x)$ iff $z_j = e^{i\theta_j}$, $z_{n+j} = \overline{z}_j$, $j = 1,\ldots,n$,$z_0 = -1$ and $z_{2n+1} = +1$ are the zeros of

$$B_{2n+1}(z,-1) = \rho_{2n+2}(z) - \rho_{2n+2}^*(z)$$

L_2-convergence

$\sigma(x)$: weight function on $[-\pi,\pi]$
$\sigma^{(0)}(x)=\sigma(x)$, $\sigma^{(1)}(x)=(1-x)\sigma(x)$, $\sigma^{(2)}(x)=(1+x)\sigma(x)$,
$\sigma^{(3)}(x)=(1-x^2)\sigma(x)$ $n\geq 1$

- $Q_n^{(l)}(x)$ $(l=0,1,2,3)$ n-th orthogonal polynomial w.r.t $\sigma_l(x)$
- $\{x_{k,n}\}_{k=1}^{n}$, the zeros of $Q_n^{(l)}(x)$, $l=0,1,2,3$.

for $k=1,\ldots,n$

1. $P_n^{(0)}(f,x)\in\Pi_{n-1}$ interpolatory $f(x)$ at $x_{k,n}^{(0)}$,
2. $P_n^{(1)}(f,x)\in\Pi_{n}$ interpolatory $f(x)$ at $x_{k,n}^{(1)}$, $x_{0,n}^{(1)}=-1$
3. $P_n^{(2)}(f,x)\in\Pi_{n}$ interpolatory $f(x)$ at $x_{k,n}^{(2)}$, $x_{0,n}^{(2)}=1$
4. $P_n^{(3)}(f,x)\in\Pi_{n+1}$ interpolatory $f(x)$ at $x_{k,n}^{(3)}$, $x_{0,n}^{(3)}=-1$, $x_{n+1,n}^{(3)}=1$

L_2-convergence

Theorem

Let $\sigma(x)$ be a weight function on $[-1,1]$ and $f(x)$ a bounded function such that $f(x)\sigma(x)$ is integrable on $[-1,1]$. Then,

$$\lim_{n\to\infty}\|f - P_n^{(l)}(f,\cdot)\|_{2,\sigma}^2 = \lim_{n\to\infty}\int_{-1}^{1}|f(x) - P_n^{(l)}(f,x)|^2\sigma(x)dx = 0$$

$l = 0, 1, 2, 3.$

L_2-convergence

$l = 0)$

$P_n^{(0)}(f,x) \in \Pi_{n-1}$: Gauss-type interpolation.(The classical Erdös-Turán Theorem)

$l = 1, 2)$

$P_n^{(l)}(f,x) \in \Pi_n$: Radau-type interpolation.

$l = 3)$

$P_n^{(3)}(f,x) \in \Pi_{n+1}$: Lobatto-type interpolation.

This Theorem was earlier proved by the authors for the particular case $\sigma(x) = \dfrac{1}{\sqrt{1-x^2}}$

An application to product integration rules

$$I_\alpha(f) = \int_{-1}^{1} f(x)\alpha(x)dx$$

$\alpha(x)$: a possibly complex function.

$$I_n^\alpha(f) = \sum_{k=1}^{n} A_k^{(\alpha)} f(x_k)$$

$\{x_k\}_1^n$ preassigned set of distinct points on $[-1,1]$

$$I_n^\alpha(f) = I_\alpha(P_{n-1}(f,\cdot)),$$

$P_{n-1}(f,x) \in \Pi_{n-1}$ s.t. $P_{n-1}(f,x_j) = f(x_j)$, $j = 1,\dots,n$

How to choose the nodes $\{x_k\}_1^n$ so that $\lim_{n\to\infty} I_n^\alpha(f) = I_\alpha(f)$?

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How to choose the nodes $\{x_k\}_1^n$ so that $\lim_{n\to\infty} I_n^\alpha(f) = I_\alpha(f)$?

The answer:

Zeros of orthogonal polynomial w.r.t an auxiliary weight function $\sigma(x)$ on $[-1, 1]$, but also including $+1, -1$ or both as nodes.

An application to product integration rules

from $\sigma(x)$ we define:
$\sigma^{(0)}(x) = \sigma(x)$, $\sigma^{(1)}(x) = (1-x)\sigma(x)$, $\sigma^{(2)}(x) = (1+x)\sigma(x)$,
and $\sigma^{(3)}(x) = (1-x^2)\sigma(x)$
for each $n \geq 1$, $x_{k,n}^{(r)}$, $k = 1, \ldots, n$ the zeros of the n-th orthogonal polynomial w.r.t $\sigma^{(r)}(x)$, $r = 0, 1, 2, 3$. Set,

1. $I_{n,0}(f) = \sum_{k=1}^{n} A_{k,n}^{(0)} f(x_{k,n}^{(0)}) = I_\alpha(f), \forall f \in \Pi_{n-1}$
2. $I_{n,1}(f) = \sum_{k=0}^{n} A_{k,n}^{(1)} f(x_{k,n}^{(1)}) = I_\alpha(f), \forall f \in \Pi_n$, $x_{0,n}^{(1)} = 1$
3. $I_{n,2}(f) = \sum_{k=0}^{n} A_{k,n}^{(2)} f(x_{k,n}^{(2)}) = I_\alpha(f), \forall f \in \Pi_n$, $x_{0,n}^{(2)} = -1$
4. $I_{n,3}(f) = \sum_{k=0}^{n+1} A_{k,n}^{(3)} f(x_{k,n}^{(3)}) = I_\alpha(f), \forall f \in \Pi_{n+1}, x_{0,n}^{(3)} = -1$ and $x_{n+1,n}^{(3)} = 1$

An application to product integration rules

Theorem

Let $\alpha(x)$ be a Lebesgue integrable and possibly complex function on $[-1,1]$ such that:

$$\int_{-1}^{1} \frac{|\alpha(x)|^2}{\sigma(x)} dx < +\infty$$

with $\sigma(x)$ a weight function on $[-1,1]$, Then, as $r = 0, 1, 2, 3.$

1. $$\lim_{n\to\infty} I_{n,r}(f) = I_\alpha(f),$$
2. $$\lim_{n\to\infty} \sum |A_{k,n}^{(r)}| f(x_{k,n}^{(r)}) = I_\alpha(f),$$

for any Riemann integrable function on $[-1,1]$.

A numerical example

$$\alpha(x) = (1-x)^a(1+x)^b,\ Re(a), Re(b) > -1$$
$$a = b = 1+i,\ \alpha(x) = (1-x^2)^{1+i},\ \sigma(x) = \frac{1}{\sqrt{1-x^2}}$$

$f(x)$	Trapezoidal Rule	Simpson Rule	Lobatto-type Rule
x^2	$.232213512391e-1$	$.222111356411e-1$	0
$\frac{1}{x^2-2}$	$.231240569930e-1$	$.210338752077e-1$	$6.28697507623e-7$
e^{x^3}	$.352497553069e-1$	$.314561179816e-1$	$3.63427207672e-7$
$\|x^2-4\|$	$.759442067263e-1$	$.671495757124e-1$	$1.8310267194e-15$
e^{e^x}	$.189040766308$	$.175162443552$	$3.63223765917e-7$
$e^x + e^{-x}$	$.745647939616e-1$	$.681138049427e-1$	$2.02839903835e-14$

CONGRATULATIONS
ADHEMAR