A matricial computation of rational quadrature formulas on the unit circle

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Conference in honor of

Adhemar Bultheel

60th birthday

Rolling Waves - Leuven, December 2008

- ★ Introduction: An approach of the question.
- ★ Matricial representation of

  orthogonal rational functions
- ★ Para-orthogonal rational functions
- ★ A computation of Szegö rational quadrature formulas
- ★ Some examples.

ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

- Measure $\operatorname{supp}\mu \subset \mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$
- Basis $\{1, z, z^2 \ldots\} \xrightarrow{\text{Gram-Schmidt}} (\varphi_n)$ OP with respect to μ
- RR $z\varphi_{n-1} = b_n\varphi_n - a_n\varphi^*_{n-1}$ $\quad |a_n| < 1 \quad b_n = \sqrt{1 - |a_n|^2}$

$$\varphi_n^*(z) = z^n\overline{\varphi_n}(z^{-1})$$

- Matrix of T_μ

$$\mathcal{H} = \begin{pmatrix} -a_1 & -b_1a_2 & -b_1b_2a_3 & \cdots \\ b_1 & -\overline{a_1}a_2 & -\overline{a_1}b_2a_3 & \cdots \\ & b_2 & -\overline{a_2}a_3 & \cdots \\ & & b_3 & \cdots \\ & & & \cdots \end{pmatrix}$$

Hessenberg
Matrix

ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

- Measure $\quad \mathrm{supp}\mu \subset \mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$
- Basis $\quad \{1, z, z^{-1}, z^2, z^{-2}, \dots\} \xrightarrow{\text{Gram-Schmidt}} (\chi_n)$ OLP wrt μ

$$(\varphi_n) \leftrightarrow (\chi_n) \quad \begin{cases} \chi_{2n} = z^{-n}\varphi_{2n}^{*} \\ \chi_{2n+1} = z^{-n}\varphi_{2n+1} \end{cases}$$

- Matrix of T_μ $\quad \mathcal{C} = \begin{pmatrix} -a_1 & -b_1 a_2 & b_1 b_2 & & & & \\ b_1 & -\overline{a_1} a_2 & \overline{a_1} b_2 & 0 & & & \\ 0 & -b_2 a_3 & -\overline{a_2} a_3 & -b_3 a_4 & b_3 b_4 & & \\ & b_2 b_3 & \overline{a_2} b_3 & -\overline{a_3} a_4 & \overline{a_3} b_4 & 0 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$

CMV Matrix

Bohnhorst, Bunse-Gerstner, Elsner, Watkins (1993), CMV (2003)

Szegö quadrature formulas

- To determine: the nodes $z_1, z_2, \ldots, z_n$ $(z_i \neq z_j, i \neq j)$ and

  the weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ $(\lambda_j > 0, j = 1, \ldots, n)$

  s.t. $$I_\mu(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta) = \sum_{j=1}^{n} \lambda_j f(z_j) = I_n(f), \quad f \in \Lambda_{p,q}, \; p+q \text{ maximum}$$

- Accuracy on $\Lambda_{-p,p} = \{z^k \,/ -p \leq k \leq p\}$, with $p \leq n-1$

- If QF exact on $\Lambda_{-(n-1),n-1}$ and $P(z) = \prod_{j=1}^{n}(z - z_j)$ then $P_n(z) = c_n \left[\varphi_n(z) + u_n \varphi_n^*(z)\right]$

- Conversely, if $P_n(z) = c_n \left[\varphi_n(z) + u_n \varphi_n^*(z)\right]$, then
  - $P_n(z)$ has exactly n diferent zeros on $\mathbb{T}$
  - Exist $\lambda_1, \lambda_2, \ldots, \lambda_n$ positive real numbers, s.t. $\boxed{I_\mu(f) = I_n(f)}$ $\forall f \in \Lambda_{-(n-1),n-1}$

W.B. Jones, O. Njåstad & W.J. Thron

NODES ↦ ZEROS OF POP

- $$z\begin{pmatrix}\varphi_0(z)\\ \vdots\\ \chi_{n-1}(z)\end{pmatrix} = \mathcal{H}_n(\boldsymbol{a};u)^T\begin{pmatrix}\varphi_0(z)\\ \vdots\\ \varphi_{n-1}(z)\end{pmatrix} + \begin{pmatrix}0\\ \vdots\\ P_n(z;u)\end{pmatrix}$$

- $$z\begin{pmatrix}\chi_0(z)\\ \vdots\\ \chi_{n-1}(z)\end{pmatrix} = C_n(\boldsymbol{a};u)^T\begin{pmatrix}\chi_0(z)\\ \vdots\\ \chi_{n-1}(z)\end{pmatrix} + b_n z^{-[\frac{n-1}{2}]}P_n(z;u)$$

$$P_n(z;u) := z\varphi_{n-1}(z) + u\varphi_{n-1}^*(z) \quad \longrightarrow \quad \text{POP}$$

NODES OF SZEGÖ QUADRATURE FORMULAS ↦
EIGENVALUES OF HESSENBERG AND CMV MATRICES

Orthogonal rational funcions

- Möbius transform $\zeta_\alpha = \dfrac{{\varpi_\alpha}^*(z)}{\varpi_\alpha(z)} = \dfrac{z-\alpha}{1-\bar{\alpha}z} \qquad \alpha \in \mathbb{D}$

- Blaschke product $B_0 = 1, \quad B_n = \zeta_1 \cdots \zeta_n = \dfrac{\pi_n^*}{\pi_n},$

$\rightsquigarrow \quad \hat{\alpha} = \{\hat{\alpha}_i, \;\; i = 1 \cdots, n\} \subset \mathbb{E}, \;\; \hat{\alpha}_i = 1/\bar{\alpha}_i, \quad \pi_n(z) = \varpi_1(z) \cdots \varpi_n(z)$

- $\mathcal{L}_n = \operatorname{span}\{B_0, B_1, \cdots B_n\} = \left\{ \dfrac{p(z)}{\pi_n(z)}; p \in P_n \right\}$
- $\mathcal{R}_{m,n} = \operatorname{span}\{B_{m*} \cdots B_{1*}, B_0, B_1, \cdots B_n\}$

- $\phi_n = \dfrac{p_n}{\pi_n} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1} \qquad \phi_n^* = B_n \phi_{n*} = \dfrac{p_n^*}{\pi_n} \in \mathcal{L}_n$

Orthogonal rational functions

- Measure $\quad \mathrm{supp}\mu \subset \mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$
- Basis $\quad \{B_0, B_1, B_2 \dots\} \xrightarrow{\text{Gram-Schmidt}} (\phi_n)$ ORF wrt μ
- RR

$$\begin{pmatrix} \phi_n(z) \\ \phi_n^*(z) \end{pmatrix} = e_n \frac{\varpi_{n-1}(z)}{\varpi_n(z)} \begin{pmatrix} 1 & a_n \\ \bar{a}_n & 1 \end{pmatrix} \begin{pmatrix} \zeta_{n-1}(z)\phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{pmatrix} \qquad n \geq 1$$

$$a_n = \frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}$$

$$e_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})} \frac{1}{1-|a_n|^2}}$$

Orthogonal rational functions - Matricial representation

- Basis of ORF (ϕ_n)
- (α_n) compactly included in $\mathbb{D}$
- Matrix of $T_\mu \restriction \mathcal{L}$ $\longrightarrow$ Operator Möbius transform of $\mathcal{H}$

$$\boxed{\nu = \zeta_{\mathcal{A}}^{-1}(\mathcal{H}) = \eta_{\mathcal{A}}^{-1}(\mathcal{H}+\mathcal{A})(1+\mathcal{A}^\dagger\mathcal{H})^{-1}\eta_{\mathcal{A}^\dagger}}$$

※ $\eta_{\mathcal{A}} = \sqrt{1-\mathcal{A}\mathcal{A}^\dagger}$ $\quad$ $\mathcal{A}^\dagger$ Adjoint of $\mathcal{A}$

※ $\mathcal{A} = \mathrm{diag}(\alpha_0, \alpha_1, \alpha_2 \cdots)$

※ $\mathcal{H}$ Hessenberg matrix

L. Velázquez, "Spectral methods for orthogonal rational functions"
Journal of Functional Analysis (2008)

O R F - A simpler matricial representation

- (ϕ_n) ORF with poles in $\mathbb{E}$
- Odd and even Blaschke products

$$B_0^o, B_0^e = 1$$

$$B_n^o = \zeta_1\zeta_3\cdots\zeta_{2n-1}$$

$$B_n^e = \zeta_2\zeta_4\cdots\zeta_{2n}$$

▶ $\chi_{2n} = B_{n*}^e\phi_{2n}^*$ $\qquad \chi_{2n+1} = B_{n*}^e\phi_{2n+1}^*$

- (χ_n) ORF with poles in $\mathbb{E}$ and $\mathbb{D}$.

▶ Matrix of T_μ wrt (χ_n) $\longrightarrow$ Operator Möbius transform of $\mathcal{C}$

$$\boxed{\mathcal{U} = \zeta_{\mathcal{A}}^{-1}(\mathcal{C})}$$

L. Velázquez, "Spectral methods for orthogonal rational functions"
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Para-orthogonal rational functions

- (ϕ_n) ORF with poles in $\mathbb{E}$
- PORF $\longrightarrow P_n^v(z) = \phi_n(z) + v\phi_n^*(z), \quad v \in \mathbb{T}$

▶ Zeros of P_n^v $\rightsquigarrow$. Eigenvalues λ of $\nu^{(n,u)}$, $u = \zeta_{\alpha_n}^{-1}(v)$

. Eigenvectors $\rightsquigarrow$ $(\phi_0(\lambda), \phi_1(\lambda), \cdots \phi_n(\lambda))^\dagger$

◇◇ λ generalized eigenvalue of the pencil $(\mathcal{A}_n + \mathcal{H}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{H}_n^u)$

▶ Zeros of P_n^v $\rightsquigarrow$. Eigenvalues λ of $\mathcal{U}^{(n,u)}$, $u = \zeta_{\alpha_n}^{-1}(v)$

. Eigenvectors $\rightsquigarrow$ $(\chi_0(\lambda), \chi_1(\lambda), \cdots \chi_n(\lambda))^\dagger$

◇◇ λ generalized eigenvalue of the pencil $(\mathcal{A}_n + \mathcal{C}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{C}_n^u)$

L. Velázquez, "Spectral methods for orthogonal rational functions"

The computation of rational Szegö quadrature formulas

- To determine: the nodes $z_1, z_2, \ldots, z_n$ $(z_i \neq z_j, i \neq j)$ and

  the weights $A_1, A_2, \ldots, A_n$ $(A_j > 0, j = 1, \ldots, n)$

s.t. $I_\mu(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta) = \sum_{j=1}^{n} \lambda_j f(z_j) = I_n(f)$, $f \in \mathcal{R}_{p,q}$, $p + q$ maximum

- $z_1, z_2, \cdots, z_N$ zeros of $P_N^{u_N}$ PORF.

  Exist $A_1, A_2, \cdots, A_N$ st $I_N\{f\} = I_\mu\{f\}$, $\forall f \in \mathcal{R}_{N-1,N-1}$

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- Nodes of rational Szegö quadrature formulas:

  Eigenvalues of matrix Möbius transform of Hessenberg

  and five-diagonal matrices

Some numerical examples

- $f(z) = \dfrac{1+z}{1-z/2} + \dfrac{z}{2-z} + \dfrac{z^2+3z-z^3}{(z-5)(1-z/6)(1-2z)}$
- Approximate $I_\mu\{f\}$, μ Lebesgue measure
- $\alpha_0 = 1$, $\alpha_n = 1/(n+1)$, $n = 7$

Generalized eigenvalue problem for the pencil $(\mathcal{A}_n + \mathcal{H}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{H}_n^u)$

$v = -1$, θ_j of nodes $z_j = e^{i\theta_j}$

Nodes	Weights
0	8.403361344537813E − 002
±2.465407008647439E + 000	2.077776600510375E − 001
±1.336738281545516E + 000	1.488205944472059E − 001
±5.650413576629920E − 001	1.013849387790675E − 001

Some numerical examples

- Lebesgue measure $\dfrac{d\theta}{2\pi}$

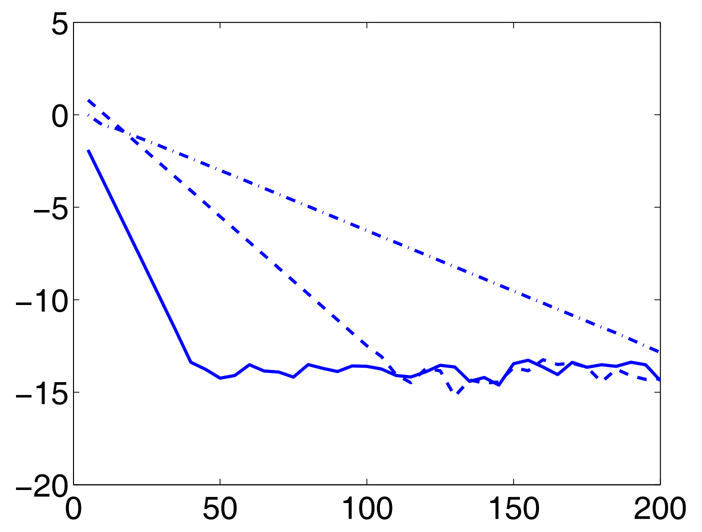
Poles $\alpha_k = \dfrac{1}{k+1}, \quad k = 1, \dots, 7, \quad v = i$

Nodes	Weights
$2.798995563470065\mathrm{E}+000$	$2.161442192462172\mathrm{E}-001$
$-2.148422117836067\mathrm{E}+000$	$1.952619467196891\mathrm{E}-001$
$1.582596667215560\mathrm{E}+000$	$1.643566429924263\mathrm{E}-001$
$-1.114446080717590\mathrm{E}+000$	$1.344496322877105\mathrm{E}-001$
$7.313150857335874\mathrm{E}-001$	$1.106238819774966\mathrm{E}-001$
$-4.118403543362309\mathrm{E}-001$	$9.399019672023695\mathrm{E}-002$
$1.325975632655727\mathrm{E}-001$	$8.517348005622358\mathrm{E}-002$

Some numerical examples

- Relative error of $\int \frac{z^p}{sin(z^5)+0.2} d\theta$

  $n = 5, 10, 15, \cdots, 200 \quad p = 0,\ p = -1,\ p = 1$



Some numerical examples

- Rational modification of Lebesgue measure

$$d\mu(\theta) = \frac{1-|r|^2}{|z-r|^2}\frac{d\theta}{2\pi}$$

- Schur parameters for the ORF $\longmapsto$ $a_1 = -r$, $a_k = 0$ $k = 2, 3\cdots$

  - $f(z) = \dfrac{1+z}{1-z/2} + \dfrac{z}{2-z} + \dfrac{z^2+3z-z^3}{(z-5)(1-z/6)(1-2z)}$,

    $r \in (-1,1)$ and differents v

  - $f(z) = \dfrac{z^p}{\sin(z^5)+0.2}$ $\quad p = -1, 0, 1$ for every $r \in \mathbb{D}$, $v \in \mathbb{T}$

- *The method converges*

Some numerical examples

- Chebyshev weights

  . $\nu \in \{-1, 1\}, \qquad d\mu(\theta) = (1 - \nu \cos\theta)\frac{d\theta}{2\pi}$

- Orthogonal Rational Functions

  . $\Phi_0 = 1,\ \Phi_n(z) = \frac{X_n(z)}{(z-\nu)^2},\ \ n = 1, 2, 3\cdots \qquad (X_n(z) = c_n + z^2(z - b_n)\frac{B_{n-1}}{1 - \bar{\alpha}z})$

- Schur parameters $\quad a_n = \rho_n^2 c_n \frac{1 - \bar{\alpha}_n \alpha_{n-1}}{1 - \bar{b}_n \alpha_{n-1}}$

- Integration of $\quad \frac{z^p}{\sin(z^5) + 0.2}$

- *The method converges*

¡ GELUKKIGE VERJAARDAG, ADHEMAR !

HAPPY BIRTHDAY, ADHEMAR !