Rolling waves in Leuven 2008

dedicated to the 60th birthday of Adhemar Bultheel

Hermite Interpolation problems and FFT

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Abstract

♦ We present a method for computing the Hermite Interpolation polynomial based on equally spaced nodes on the unit circle with an arbitrary number of derivatives in the case of algebraic and Laurent polynomials.

♦ Indeed we present a smart adaptation of the method of the Fast Fourier transform (FFT) for this type of problems with the following characteristics:

- easy computation,
- small number of operations,
- easy implementation.

♦ We adapt the algorithm for computing the Hermite Interpolation polynomial based on the nodes of the Chebyshev polynomials.

♦ We also study Hermite trigonometric interpolation problems.

1. Introduction and some definitions.

We study some interpolation problems on the unit circle, corresponding to the case of equally spaced nodes. For simplicity and without loss of generality we assume that the nodes are the n roots of the unity $\{z_j\}_{j=0}^{n-1}$.

If we fix a vector $\{u_j\}_{j=0}^{n-1}$, the **general interpolation problem** is to determine a polynomial $p_{n-1}(z)$ with degree less than or equal to $n-1$ such that

$$p_{n-1}(z) \in \mathbb{P}_{n-1}[z], \;\; p_{n-1}(z_j) = u_j, \quad \text{for } j = 0, \cdots, n-1. \tag{1}$$

It is well-known that this polynomial exists and it is unique.

Among the different methods for computing $p_{n-1}(z)$ it may be pointed up the method which uses the Fast Fourier Transform (FFT) for obtaining its coefficients.

Indeed $p_{n-1}(z) = \sum_{k=0}^{n-1} \tilde{c}_k z^k$ with $\tilde{c}_k = \frac{1}{n} \sum_{j=0}^{n-1} u_j \overline{z_j}^k$, that can be computed by using the FFT (assuming that n is a power of 2).

Extending this method to the Hermite interpolation problem

Definition 1 *We call Hermite interpolation problem with equally spaced nodes* $\{z_j\}_{j=0}^{n-1}$ *on* $\mathbb{T}$ *to determine a polynomial*

$$H_{2n-1}(z) \in \mathbb{P}_{2n-1}[z] \;:\; H_{2n-1}(z_j) = u_j, \quad H_{2n-1}^{(1)}(z_j) = v_j \quad \textit{for } j = 0, \cdots, n-1,$$

where $\{u_j\}_{j=0}^{n-1}$ *and* $\{v_j\}_{j=0}^{n-1}$ *are fixed vectors.*

It is well-known that this polynomial exists and it is unique. Moreover, it is given by:

$$H_{2n-1}(z) = \sum_{k=0}^{n-1} A_k(z) u_k + \sum_{k=0}^{n-1} B_k(z) v_k, \tag{2}$$

with $A_k(z) = \left(1 - 2(z - z_k) l_k^{(1)}(z)\right) l_k^2(z)$, $B_k(z) = (z - z_k) l_k^2(z)$, $k = 0, \cdots, n-1$, where $l_k(z)$ are called the fundamental polynomials of Lagrange interpolation and they are given by $l_k(z) = \frac{W_n(z)}{W_n^{(1)}(z_k)(z-z_k)}$, for $k = 0, \cdots, n-1$, with $W_n(z) = \Pi_{k=0}^{n-1}(z - z_k) = z^n - 1$.

Definition 2 *We call Hermite interpolation problem of type I with equally spaced nodes* $\{z_j\}_{j=0}^{n-1}$ *on the unit circle* $\mathbb{T}$ *to determine a polynomial* $HI_{2n-1}(z) \in \mathbb{P}_{2n-1}[z]$ *such that*

$$HI_{2n-1}(z_j) = u_j \quad \text{and} \quad HI_{2n-1}^{(1)}(z_j) = 0 \quad \text{for } j = 0, \cdots, n-1.$$

Definition 3 *We call Hermite interpolation problem of type II with equally spaced nodes* $\{z_j\}_{j=0}^{n-1}$ *on the unit circle* $\mathbb{T}$ *to determine a polynomial* $HII_{2n-1}(z) \in \mathbb{P}_{2n-1}[z]$ *such that*

$$HII_{2n-1}(z_j) = 0 \quad \text{and} \quad HII_{2n-1}^{(1)}(z_j) = v_j \quad \text{for } j = 0, \cdots, n-1.$$

From the general solution (2) we obtain the explicit solutions

$$HI_{2n-1}(z) = \sum_{k=0}^{n-1} \left(\frac{z_k^2}{n^2} \frac{(z^n - 1)^2}{(z - z_k)^2} u_k - \frac{z_k(n-1)}{n^2} \frac{(z^n - 1)^2}{(z - z_k)} u_k \right),$$

and

$$HII_{2n-1}(z) = \sum_{k=0}^{n-1} \frac{z_k^2}{n^2} \frac{(z^n - 1)^2}{(z - z_k)} v_k.$$

Next we are going to present an algorithm using the FFT to compute $HI_{2n-1}(z)$ and $HII_{2n-1}(z)$.

2. FFT for solving Hermite interpolation problems on $\mathbb{T}$.

We consider the inner product of Sobolev type in the space $\mathbb{P}_{2n-1}[z]$, associated with the system of equally spaced nodes $\{z_j\}_{j=0}^{n-1}$, defined by

$$\langle f(z), g(z)\rangle_s = \frac{1}{n}\sum_{j=0}^{n-1} f(z_j)\overline{g(z_j)} + \frac{1}{n}\sum_{j=0}^{n-1} f^{(1)}(z_j)\overline{g^{(1)}(z_j)}.$$

Theorem 1 *The following system*

$$\{Z_k(z)\}_{k=0}^{2n-1} = \left\{z^k\right\}_{k=0}^{n-1} \bigcup \left\{z^{n+k} - \frac{1+(n+k)k}{1+k^2} z^k\right\}_{k=0}^{n-1}$$

is an ***orthogonal basis*** *of the space $\mathbb{P}_{2n-1}[z]$ with the inner product $\langle , \rangle_s$ and the norms are*

$$\|Z_k(z)\|_s = \left(1+k^2\right)^{\frac{1}{2}},\ k = 0, \cdots, n-1,$$

$$\|Z_{n+k}(z)\|_s = \left(\frac{n^2}{1+k^2}\right)^{\frac{1}{2}},\ k = 0, \cdots, n-1.$$

2.1. Solution of the Hermite interpolation problem of type I.

Theorem 2 *It holds that* $HI_{2n-1}(z) = \sum_{k=0}^{2n-1} c_k Z_k(z)$, *where*

$$c_k = \frac{1}{1+k^2}\,\frac{1}{n}\sum_{j=0}^{n-1} u_j \overline{z_j}^k, \quad c_{n+k} = \frac{-k}{n}\,\frac{1}{n}\sum_{j=0}^{n-1} u_j \overline{z_j}^k, \ (k=0,\ \cdots,\ n-1).$$

Proof. $$\langle HI_{2n-1}(z), Z_k(z)\rangle_s = c_k \|Z_k\|_s^2 = c_k(1+k^2),$$

$$\langle HI_{2n-1}(z), Z_k(z)\rangle_s = \frac{1}{n}\sum_{j=0}^{n-1} HI_{2n-1}(z_j)\overline{z_j}^k = \frac{1}{n}\sum_{j=0}^{n-1} u_j \overline{z_j}^k.$$

$$\langle HI_{2n-1}(z), Z_{n+k}(z)\rangle_s = c_{n+k}\|Z_{n+k}\|_s^2 = c_{n+k}\frac{n^2}{(1+k^2)},$$

$$\langle HI_{2n-1}(z), Z_{n+k}(z)\rangle_s = \frac{1}{n}\sum_{j=0}^{n-1} u_j \left(\overline{z_j}^{n+k} - \frac{1+(n+k)k}{1+k^2}\overline{z_j}^k\right)$$

$$= \left(1 - \frac{1+(n+k)k}{1+k^2}\right)\frac{1}{n}\sum_{j=0}^{n-1} u_j \overline{z_j}^k.$$

Remark 1 *(i) The coefficients* c_k *that determine the solution of the Hermite problem of type I can be related with the coefficients* $\tilde{c}_k = \frac{1}{n}\sum_{j=0}^{n-1} u_j \overline{z_j}^k$ *of the polynomial* $p_{n-1}(z)$ *solution of the general interpolation problem posed in (1). Indeed*

$$c_k = \frac{1}{1+k^2}\tilde{c}_k, \quad \text{and } c_{n+k} = -\frac{k}{n}\tilde{c}_k,$$

for $k = 0, \cdots, n-1$ *and therefore they can be obtained using the FFT with a little increase of operations.*

(ii)

$$HI_{2n-1}(z) = \sum_{k=0}^{n-1} \left(c_k - c_{n+k}\frac{1+(n+k)k}{1+k^2} \right) z^k + \sum_{k=0}^{n-1} c_{n+k} z^{n+k}.$$

(iii) The ***operational cost of the method*** *is* $O(n(\log n + 1))$ *operations for the determination of the polynomial* $HI_{2n-1}(z)$ *that satisfies* $2n$ *interpolation conditions.*

2.2. Solution of the Hermite interpolation problem of type II.

Theorem 3 *It holds that*

$$HII_{2n-1}(z) = \sum_{k=0}^{2n-1} d_k Z_k(z),$$

where d_k are given by

$$d_k = \frac{k}{1+k^2}\ \frac{1}{n}\sum_{j=0}^{n-1} v_j \overline{z_j}^{k-1} = \frac{k}{1+k^2}\frac{1}{n}\sum_{j=0}^{n-1} \frac{v_j}{\overline{z_j}}\overline{z_j}^{k}$$

$$d_{n+k} = \frac{1}{n}\ \frac{1}{n}\sum_{j=0}^{n-1} v_j \overline{z_j}^{k-1} = \frac{1}{n}\frac{1}{n}\sum_{j=0}^{n-1} \frac{v_j}{\overline{z_j}}\overline{z_j}^{k}$$

for $k = 0, \cdots, n-1$.

Remark 2 *(i) The coefficients* d_k *that determine the solution of the Hermite interpolation problem of type II depend on the values* $\frac{1}{n}\sum_{j=0}^{n-1}\frac{v_j}{\overline{z_j}}\overline{z_j}^k$, *which are the coefficients* $\tilde{d}_k$ *solution of the general interpolation problem (1) with values* $\frac{v_j}{\overline{z_j}}$. *Then every method used to determine* $\tilde{d}_k$ *is useful to determine* d_k *with a little increase of operations.*

(ii)

$$HII_{2n-1}(z) = \sum_{k=0}^{n-1}\left(d_k - d_{n+k}\frac{1+(n+k)k}{1+k^2}\right)z^k + \sum_{k=0}^{n-1} d_{n+k} z^{n+k}.$$

(iii) The ***operational cost of the method*** *is* $O(n(\log n + 1))$ *operations for the determination of the polynomial* $HII_{2n-1}(z)$ *that satisfies* $2n$ *interpolation conditions.*

2.3. Solution of the Hermite interpolation problem.

Theorem 4 *The polynomial $H_{2n-1}(z)$ solution of the Hermite interpolation problem is given by*

$$H_{2n-1}(z) = \sum_{k=0}^{2n-1} (c_k + d_k) Z_k(z),$$

where $\{c_k\}$ and $\{d_k\}$ are given in Theorems 2 and 3, respectively, that is, as solutions of ***Hermite interpolation problems I and II****.*

Remark 3 *(i) The polynomial $H_{2n-1}(z)$ can be written:*

$$H_{2n-1}(z) = \sum_{k=0}^{n-1} \left(c_k + d_k - (c_{n+k} + d_{n+k}) \frac{1 + (n+k)k}{1 + k^2} \right) z^k + \sum_{k=0}^{n-1} (c_{n+k} + d_{n+k}) z^{n+k}.$$

(ii) The Hermite interpolation problem can be solved using the algorithm of the FFT with an ***operational cost*** *of $O(n(\log n + 1))$ operations.*

3. The general Hermite interpolation problem on $\mathbb{T}$.

For simplicity we continue assuming that the nodes are the n-roots of the unity $\{z_j\}_{j=0}^{n-1}$.

Let N be a positive integer $N \geq 1$, and let $\{u_0^{i)}, \cdots, u_{n-1}^{i)}\}_{i=0,\cdots,N-1}$ be fixed values. The problem is to determine a polynomial $H_{Nn-1}(z) \in \mathbb{P}_{Nn-1}[z]$ such that

$$H_{Nn-1}^{(i)}(z_j) = u_j^{i)} \text{ for } j = 0, \cdots, n-1, \; i = 0, \cdots, N-1.$$

It is well-known that this polynomial exists and it is unique.
For the obtention we can decompose the problem into l different problems with $0 \leq l \leq N-1$ in the following way:

Find the polynomial $Hl_{Nn-1}(z)$ such that
$Hl_{Nn-1}^{(i)}(z_j) = 0$ for $j = 0, \cdots, n-1$, $i = 0, \cdots, N-1$ and $i \neq l$ and $Hl_{Nn-1}^{(l)}(z_j) = u_j^{l)}$ for $j = 0, \cdots, n-1$.

Hence $H_{Nn-1}(z) = \sum_{l=0}^{N-1} Hl_{Nn-1}(z)$.

In the space $\mathbb{P}_{Nn-1}[z]$ we consider the inner product of Sobolev type

$$\langle f(z), g(z)\rangle_{s_N} = \frac{1}{n}\sum_{j=0}^{n-1} f(z_j)\overline{g(z_j)} + \cdots + \frac{1}{n}\sum_{j=0}^{n-1} f^{(N-1)}(z_j)\overline{g^{(N-1)}(z_j)}.$$

and we denote by $\{F_k(z)\}_{k=0}^{Nn-1}$ an orthogonal basis of $\mathbb{P}_{Nn-1}[z]$. Then

$$Hl_{Nn-1}(z) = \sum_{k=0}^{Nn-1} c_{k,l} F_k(z), \text{ with } c_{k,l} = \frac{1}{n}\frac{1}{\| F_k \|_{s_N}^2}\sum_{j=0}^{n-1} u_j^{l)}\overline{F_k^{l)}}(z_j).$$

It is easy to prove that

$$F_{ln+k}(z) = z^{ln+k} + a_{(l-1)n+k} z^{(l-1)n+k} + \cdots + a_{n+k} z^{n+k} + a_k z^k,$$

for $k = 0, \cdots, n-1$ and $l = 0, \cdots, N-1$.
We can obtain the coefficients $a_{(l-1)n+k}, \cdots, a_{n+k}, a_k$ solving the system :

$$\langle F_{ln+k}(z), z^{(l-1)n+k}\rangle_{s_N} = \cdots = \langle F_{ln+k}(z), z^k\rangle_{s_N} = 0,$$

and we can compute $\| F_{ln+k}(z) \|_{s_N}^2$.
Then we can determine the coefficients $c_{k,l}$. Notice that they are closely related with the values $\frac{1}{n}\sum_{j=0}^{n-1} \frac{u_j^{l)}}{\overline{z_j}^l}\overline{z_j}^k$.

4. Hermite interpolation problem on $\mathbb{T}$ in the space Λ.

Theorem 5 [a] *Let p and q be two nondecreasing sequences of nonnegative integers such that $p+q=2n-1,\ n\geq 1$. Then there exists a unique polynomial L in the Laurent space $L\in\Lambda_{-p,q}=span\{z^k : -p\leq k\leq q\}$ such that $L(z_k)=u_k,\ L^{(1)}(z_k)=v_k,\ k=0,\cdots,n-1$, and L has the following expression*

$$L(z)=\sum_{k=0}^{n-1}A_k^*(z)u_k+\sum_{k=0}^{n-1}B_k^*(z)v_k,$$

with $A_k^*(z)=\frac{z_k^{p+2}(z^n-1)^2}{z^p n^2(z-z_k)^2}+\frac{(p-n+1)z_k^{p+1}(z^n-1)^2}{z^p n^2(z-z_k)}$ *and* $B_k^*(z)=\frac{z_k^{p+2}(z^n-1)^2}{z^p n^2(z-z_k)}$.

In particular we have the following problems of type I and II respectively:

Find $L_1\in\Lambda_{-p,q}$ such that $L_1(z_k)=u_k,\ L_1^{(1)}(z_k)=0,\ k=0,\cdots,n-1$.

Find $L_2\in\Lambda_{-p,q}$ such that $L_2(z_k)=0,\ L_2^{(1)}(z_k)=v_k,\ k=0,\cdots,n-1$.

[a] L. Daruis, P. González-Vera, "A note on Hermite-Fejér interpolation for the unit circle", Appl. Math. Letters 14 (2001), 997-1003.

Theorem 6 *$L_1 \in \Lambda_{-p,q}$ such that $L_1(z_k) = u_k,\ L_1^{(1)}(z_k) = 0,\quad k = 0, \cdots, n-1$ is given by*

$$L_1(z) = \sum_{k=0}^{2n-1} (\tilde{c}_k + \tilde{d}_k) \frac{Z_k(z)}{z^p}, \qquad \textit{with}$$

$$\tilde{c}_k = \frac{1}{1+k^2} \frac{1}{n} \sum_{j=0}^{n-1} \frac{u_j}{\overline{z_j}^p} \overline{z_j}^k,\ \ \tilde{c}_{n+k} = -\frac{k}{n} \frac{1}{n} \sum_{j=0}^{n-1} \frac{u_j}{\overline{z_j}^p} \overline{z_j}^k,$$

$$\tilde{d}_k = \frac{kp}{1+k^2} \frac{1}{n} \sum_{j=0}^{n-1} \frac{u_j}{\overline{z_j}^p} \overline{z_j}^k,\ \ \tilde{d}_{n+k} = \frac{p}{n} \frac{1}{n} \sum_{j=0}^{n-1} \frac{u_j}{\overline{z_j}^p} \overline{z_j}^k.$$

$L_2 \in \Lambda_{-p,q}$ such that $L_2(z_k) = 0,\ L_2^{(1)}(z_k) = v_k,\ k = 0, \cdots, n-1$ is given

$$L_2(z) = \sum_{k=0}^{2n-1} \hat{d}_k \frac{Z_k(z)}{z^p}, \qquad \textit{with}$$

$$\hat{d}_k = \frac{k}{1+k^2} \frac{1}{n} \sum_{j=0}^{n-1} \frac{v_j}{\overline{z_j}^{p+1}} \overline{z_j}^k,\ \ \hat{d}_{n+k} = \frac{1}{n} \frac{1}{n} \sum_{j=0}^{n-1} \frac{v_j}{z_j^{p+1}} \overline{z_j}^k.$$

5. Algorithms for solving:

5.1. Hermite interpolation on the interval $[-1, 1]$.

Let $\{x_j\}_{j=0}^{n-1}$ be the n **roots of the Tchebychef polynomial of the first kind** $T_n(x)$, that is, $x_j = \cos\left(\frac{(2j+1)\pi}{2n}\right)$, $j = 0, \cdots, n-1$.

Let us consider the Hermite interpolation **problem**:

Find a polynomial $h_{2n-1}(x) \in \mathbb{P}_{2n-1}[x]$ such that

$$h_{2n-1}(x_j) = m_j,\ h^{(1)}_{2n-1}(x_j) = n_j,\ j = 0, \cdots, n-1,$$

where $\{m_j\}_{j=0}^{n-1}$ and $\{n_j\}_{j=0}^{n-1}$ are fixed values.

This problem has unique solution and it is well-known its solution, [Da][a]
Thus, our aim is to give an alternative algorithm to compute it.

[a] P.J. Davis, "Interpolation & Approximation", Dover Pub. (1975).

Let $w_j = e^{i\frac{(2j+1)\pi}{2n}}$, $j = 0, \cdots, n-1$.
It is immediate that $w_j^{2n} = -1$, $j = 0, \cdots, n-1$ and therefore $\{w_j, \overline{w}_j\}_{j=0}^{n-1}$ are the **2n-roots** of **-1**, and $\frac{w_j + \overline{w}_j}{2} = x_j$, $j = 0, \cdots, n-1$.

Problem. Find the Laurent polynomial $\mathcal{L} \in \Lambda_{-(2n-1),2n}$ such that

$$\mathcal{L}(w_j) = \mathcal{L}(\overline{w}_j) = m_j, \; j = 0, \cdots, n-1,$$
$$\mathcal{L}^{(1)}(w_j) = n_j\sqrt{1 - x_j^2}\,\overline{w}_j\imath, \; \mathcal{L}^{(1)}(\overline{w}_j) = -n_j\sqrt{1 - x_j^2}\,w_j\imath.$$

So we have to solve in the space of Laurent polynomials an interpolation problem on the unit circle with nodes the $2n$ roots of -1.

Since we have studied the case corresponding to the roots of 1, first we have reformulated the results obtained before for this new situation, indeed for a more general situation, the n-roots of λ, with $|\lambda| = 1$.

- The Laurent polynomial $\mathcal{L} \in \Lambda_{-(2n-1),2n}$ such that

$$\mathcal{L}(w_j) = \mathcal{L}(\overline{w}_j) = m_j, \; j = 0, \cdots, n-1,$$
$$\mathcal{L}^{(1)}(w_j) = n_j\sqrt{1-x_j^2}\,\overline{w}_j\imath, \; \mathcal{L}^{(1)}(\overline{w}_j) = -n_j\sqrt{1-x_j^2}\,w_j\imath.$$

is given by

$$\mathcal{L}(z) = \sum_{k=0}^{2n-1}[-\frac{k+1}{4n^2}\sum_{j=0}^{n-1} m_j(\overline{w}_j^{k+1} + w_j^{k+1}) +$$
$$\frac{\imath}{4n^2}\sum_{j=0}^{n-1} n_j\sqrt{1-x_j^2}(\overline{w}_j^{k+1} - w_j^{k+1})]\frac{1}{z^{2n-k-1}} +$$
$$\sum_{k=0}^{2n-1}[\frac{-k+2n-1}{4n^2}\sum_{j=0}^{n-1} m_j(\overline{w}_j^{k+1} + w_j^{k+1}) + \frac{\imath}{4n^2}\sum_{j=0}^{n-1} n_j\sqrt{1-x_j^2}(\overline{w}_j^{k+1} - w_j^{k+1})]z^{k+1}$$

- Indeed $\mathcal{L}(z) \in \Lambda_{-(2n-1),2n-1}$ and it is symmetric in z^j and $\frac{1}{z^j}$.
- The Laurent polynomial $\mathcal{L}(z)$ can be written as follows

$$\mathcal{L}(z) = \frac{1}{n}\sum_{j=0}^{n-1} m_j + \sum_{k=1}^{2n-1}\left(\frac{(2n-k)}{4n^2}\sum_{j=0}^{n-1} m_j(\overline{w}_j^k + w_j^k) + \frac{\imath}{4n^2}\sum_{j=0}^{n-1} n_j\sqrt{1-x_j^2}(\overline{w}_j^k - w_j^k)\right)(z^k + \frac{1}{z^k}) \tag{3}$$

- For computing $\mathcal{L}(z)$ we have to compute $\sum_{j=0}^{n-1} m_j(\overline{w}_j^k + w_j^k)$ and $\sum_{j=0}^{n-1} n_j\sqrt{1-x_j^2}(\overline{w}_j^k - w_j^k)$.

Notice that they can be determined by computing $e^{i\frac{k\pi}{2n}}\sum_{j=0}^{2n-1} M_j z_j^k$, with $M_j = m_j$ for $0 \le j \le n-1$ and $M_{j+n} = m_{n-j-1}$ for $0 \le j \le n-1$ and $e^{i\frac{k\pi}{2n}}\sum_{j=0}^{2n-1} N_j z_j^k$, with $N_j = -n_j\sqrt{1-x_j^2}$ for $0 \le j \le n-1$ and $N_{j+n} = n_{n-j-1}\sqrt{1-x_{n-j-1}^2}$ for $0 \le j \le n-1$.

If we define $h_{2n-1}(x) = \mathcal{L}(z)$ with $x = \dfrac{z + 1/z}{2}$ and $\mathcal{L}(z)$ given by (3), we obtain.

Theorem 7 *The polynomial* $h_{2n-1}(x) \in \mathbb{P}_{2n-1}[x]$ *such that*

$$h_{2n-1}(x_j) = m_j,\ h^{(1)}_{2n-1}(x_j) = n_j,\ j = 0, \cdots, n-1,$$

where $\{m_j\}_{j=0}^{n-1}$ *and* $\{n_j\}_{j=0}^{n-1}$ *are fixed values is given by*

$$h_{2n-1}(x) = \frac{1}{n}\sum_{j=0}^{n-1} m_j +$$

$$\sum_{k=1}^{2n-1}\left(\frac{(2n-k)}{2n^2}\sum_{j=0}^{n-1} m_j(\overline{w}_j^k + w_j^k) + \frac{\imath}{2n^2}\sum_{j=0}^{n-1} n_j\sqrt{1-x_j^2}(\overline{w}_j^k - w_j^k)\right)T_n(x).$$

• For evaluating $h_{2n-1}(x)$ at a point $\tilde{x}$ we evaluate $\mathcal{L}(z)$ given by (3) at $\tilde{z}$ such that $\tilde{x} = \dfrac{\tilde{z} + 1/\tilde{z}}{2}$.

5.2. Hermite trigonometric interpolation on $[0, 2\pi]$.

We present the formulae for the coefficients of the Hermite interpolant. Although our approach is different, the results can be seen in [BW] [a].

The **problem** that we want to solve is the following:

Let us consider the equidistant nodes $\{\theta_j\}_{j=0}^{n-1}$ in $[0, 2\pi)$ with $\theta_j = \dfrac{2j\pi}{n}$ for $j = 0, \cdots, n-1$.

Find a trigonometric polynomial $T(\theta)$ with real coefficients and degree at most n such that

$$T(\theta_j) = m_j \quad \text{and} \quad T^{(1)}(\theta_j) = n_j \quad \text{for } j = 0, \cdots, n-1,$$

where $\{m_j\}_{j=0}^{n-1}$ and $\{n_j\}_{j=0}^{n-1}$ are fixed values.

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[a] J. P. Berrut, A. Welscher, "Fourier and bary centric formulae for equidistant Hermite trigonometric interpolation", Appl. Comput. Harmon. Anal. 23 (2007), 307-320.

For solving this problem we solve the following **Hermite interpolation problem on** $\mathbb{T}$:

Find the Laurent polynomial $L \in \Lambda_{-(n-1),n}$ such that $L(z_j) = m_j$ and $L^{(1)}(z_j) = -\imath \overline{z_j} n_j$ where $z_j = e^{\imath\theta_j}$ for $j = 0, \cdots, n-1$.

Since $\{z_j\}_{j=0}^{n-1}$ are the n **roots of** 1, we apply the preceding results to obtain L.

After doing some easy computations we have:

$$L(z) = -\frac{\imath}{n^2}\sum_{j=0}^{n-1} n_j z^n + \sum_{k=1}^{n-1}\left(\left(\frac{n-k}{n^2}\right)\sum_{j=0}^{n-1} m_j \overline{z_j}^k - \frac{\imath}{n^2}\sum_{j=0}^{n-1} n_j \overline{z_j}^k\right) z^k +$$
$$\frac{1}{n}\sum_{j=0}^{n-1} m_j + \frac{\imath}{n^2}\sum_{j=0}^{n-1} n_j + \sum_{k=1}^{n-1}\left(\left(\frac{n-k}{n^2}\right)\sum_{j=0}^{n-1} m_j z_j^k + \frac{\imath}{n^2}\sum_{j=0}^{n-1} n_j z_j^k\right)\frac{1}{z^k}.$$

Now, if we define $T(\theta) = L(e^{\imath\theta})$ we obtain that $T(\theta)$ is a trigonometric polynomial with complex coefficients, degree at most n and it satisfies the interpolation conditions.

Let us write $L(e^{\imath\theta}) = \Re L(e^{\imath\theta}) + \imath \Im L(e^{\imath\theta})$. Then

$$T_1(\theta) = \Re L(e^{\imath\theta}) = \frac{1}{n^2}\sum_{j=0}^{n-1} n_j \sin n\theta +$$
$$2\left(\sum_{k=1}^{n-1}(\frac{n-k}{n^2})\Re(\sum_{j=0}^{n-1} m_j z_j^k) - \frac{1}{n^2}\Im(\sum_{j=0}^{n-1} n_j z_j^k)\right)\cos k\theta -$$
$$2\left(\sum_{k=1}^{n-1}(\frac{n-k}{n^2})\Im(\sum_{j=0}^{n-1} m_j z_j^k) + \frac{1}{n^2}\Re(\sum_{j=0}^{n-1} n_j z_j^k)\right)\sin k\theta + \frac{1}{n}\sum_{j=0}^{n-1} m_j.$$

$$T_2(\theta) = \Im L(e^{\imath\theta}) = -\frac{1}{n^2}\sum_{j=0}^{n-1} n_j \cos n\theta + \frac{1}{n^2}\sum_{j=0}^{n-1} n_j.$$

Since $T_2(\theta_j) = T_2^{(1)}(\theta_j) = 0$ for $j = 0, \cdots, n-1$ then $T_1(\theta)$ fulfills the conditions and it has real coefficients.
Therefore $T_1(\theta)$ is **a solution of our problem**, [a]. Notice that its coefficients depend on $\sum_{j=0}^{n-1} m_j z_j^k$ and $\sum_{j=0}^{n-1} n_j z_j^k$.

[a] R. Kress, "On general Hermite trigonometric interpolation", Numer. Math. 20 (1972), 125-138.

6. Final remarks.

• For solving the Hermite interpolation problem on $\mathbb{T}$, with n nodes, in the space of Laurent polynomials we need to compute the following sums:

$$\sum_{j=0}^{n-1} \frac{u_j}{\overline{z_j}^p} \overline{z_j}^k \quad \text{and} \quad \sum_{j=0}^{n-1} \frac{v_j}{\overline{z_j}^{p+1}} \overline{z_j}^k.$$

Therefore, by doing 6 multiplications we obtain the coefficients:

$$\tilde{c}_k, \tilde{c}_{n+k}, \tilde{d}_k, \tilde{d}_{n+k}, \hat{d}_k, \hat{d}_{n+k},$$

and, if FFT could be applied we need $O(n(logn + 1))$ operations to obtain all the coefficients.
Notice that the different problems studied:

1. The Hermite interpolation problem on $\mathbb{T}$, with n nodes, in the space of algebraic polynomials,
2. The Hermite interpolation on the interval $[-1, 1]$,
3. The Hermite trigonometric interpolation on $[0, 2\pi]$,

can be solved from the solution of the preceding problem by setting the suitable interpolation conditions.

- The Hermite interpolation problem on $\mathbb{T}$ in the space of Laurent polynomials,

- The Hermite interpolation on the interval $[-1, 1]$ and

- The Hermite trigonometric interpolation problem on $[0, 2\pi]$

can be formulated with **higher derivatives, not** necessarily **consecutives**.

The solution follows from the case of **algebraic polynomials**.