The Tile Packing Problem

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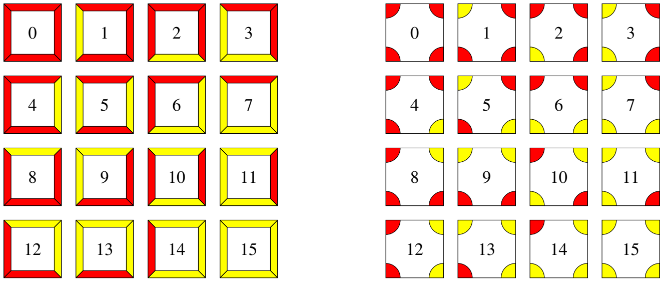
Abstract

The tile packing problem is a challenging combinatorial puzzle based on tiles with colored edges or colored corners. In its different incarnations, the puzzle gives rise to a number of interesting problems. In this paper, we sketch the background of the tile packing problem and present solutions to the puzzle. We hope that this work will stimulate further interest in this puzzle amongst readers, and that the remaining open problems will eventually be solved.

1 Introduction

In this paper we discuss the *tile packing* problem. This problem involves two kinds of tiles: *edge tiles* and *corner tiles*. Edge tiles are square tiles with colored edges, and corner tiles are square tiles with colored corners. Edge tiles and corner tiles have a fixed orientation, they may not be rotated. A *tile set* is a finite set of tiles. A *complete tile set* contains a tile for every possible combination of four colors. A complete set of edge tiles or corner tiles over C colors counts C^4 tiles. Figure 1 shows the complete edge and corner tile sets over 2 colors. The tile packing problem



(a) (b)

Figure 1: The complete set of (a) edge tiles and (b) corner tiles over 2 colors.

consists of arranging a complete set of C^4 edge tiles or corner tiles in a $C^2 \times C^2$ toroidal configuration such that adjoining edges or corners have matching colors. Edge tiles and corner tiles easily generalize to arbitrary dimensions, yielding a family of related problems.

This paper is structured as follows. Section 2 discusses the background of the tile packing problem. In section 3 we show how to solve the edge tile packing problem, and in section 4 we attempt to solve the corner tile packing problem. We present puzzles derived from the tile packing problem in section 5. In section 6 we conclude.

2 Background

Edge tiles are also known as *Wang tiles*. Wang tiles were first proposed by Hao Wang in 1961 [Wan61], and later popularized in an article in *Scientific American* [Wan65]. In his paper, Wang presented an algorithm to decide whether a given set of Wang tiles could tile the plane. He relied on the conjecture that *aperiodic tile sets*, tile sets that do not admit periodic tilings, do not exist.

This conjecture was in 1966 refuted by Berger [Ber66]. He showed that any Turing machine can be translated into a Wang tile set, and that the Wang tile set tiles the plane if and only if the Turing machine will never

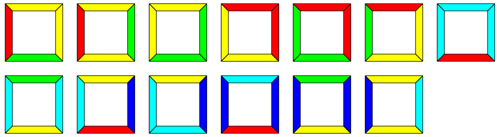


Figure 2: An aperiodic Wang tile set of 13 tiles over 5 colors.

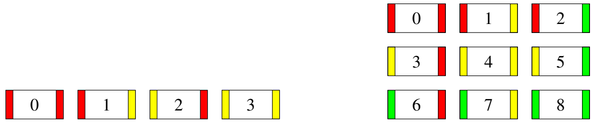
halt. The halting problem is undecidable and thus so is Wang's original problem.

Berger constructed the first aperiodic tile set counting 20426 tiles. This number was reduced repeatedly, often by well known scientists, such as Donald Knuth [Knu68]. The smallest aperiodic set of Wang tiles consists of 13 tiles over 5 colors [Cul96], and is shown in figure 2. Not only Wang tiles allow the construction of aperiodic sets. In 1974, Roger Penrose discovered his aperiodic set of only two tiles [Pen74]. Whether a single aperiodic tile exist is still an open question.

Wang tiles have several practical applications in the field of Computer Graphics [CSHD03, Wei04]. Recently, we have introduced tiles with colored corners and we have shown that corner tiles are better suited for these applications [LD06a]. Tile-based texture mapping [Wei04, LD06a], one of these applications, requires the tiles to be packed together into a toroidal configuration. This is the origin of the tile packing problem.

We have shown recently that small aperiodic sets of corner tiles exist [LKD06]. The smallest set of aperiodic corner tiles we constructed consists of 44 tiles over 6 colors. For more information about (aperiodic) tilings, we refer the interested reader to [GS86].

The work most closely related to the tile packing problem in the field of recreational mathematics is that of Major P A MacMahon [Mac21]. He describes sets of pieces of different geometrical forms (equilateral triangles, squares, pentagons, etc.) with colored edges that are tiled into another geometrical form. The profile of the adjoining edges is then altered to produce jigsaw puzzles. His work is unique in the fact that it details how the puzzles can be constructed and solved. In contrast with edge and corner tiles, MacMahon's pieces may be rotated, and MacMahon does not consider pieces with colored corners.



(a) (b)

Figure 3: The one-dimensional complete set of Wang or corner tiles over (a) 2 and (b) 3 colors.

3 The Edge Tile Packing Problem

In one dimension, edge tiles and corner tiles can be seen as *dominoes*. a complete set of Wang or corner tiles over C colors counts C^2 tiles. Figure 3 shows the complete set of Wang or corner tiles over 2 and 3 colors. The tile packing problem consists of arranging a complete set of C^2 tiles into a single circular train.

Domino problems like this one are well known in the field of recreational mathematics, and can easily be solved using graph theory [Bal26]. The tile set is represented as a directed graph with a vertex for each color, and an edge connecting two vertices for each tile in the tile set. Figure 4 shows the graphs for the complete tile sets over 2 and 3 colors. A solution for the tile packing problem is given by an *Eulerian circuit*, a graph cycle that uses each graph edge exactly once. A complete tile set results in a *complete directed graph*, which always has an Eulerian circuit. Figure 5 shows tile packings of the complete tile sets over 2 and 3 colors.

Wei observed that a solution for the two-dimensional tile packing problem is given by the *outer product* of two one-dimensional tile packings [Wei04]. This procedure is illustrated in figure 6. A one-dimensional tile packing of a complete set over C colors consist of C^2 tiles. The outer product of two such tile packings produces a matrix of C^4 tiles. Because adjoining edges have matching colors, and each tile occurs once, this is a tile packing of the C^4 tiles of a complete tile set over C colors. This construction method generalizes to tile packings of edge tiles in any dimension. We call an edge tile packing obtained with this construction method an *Eulerian edge tile packing*.

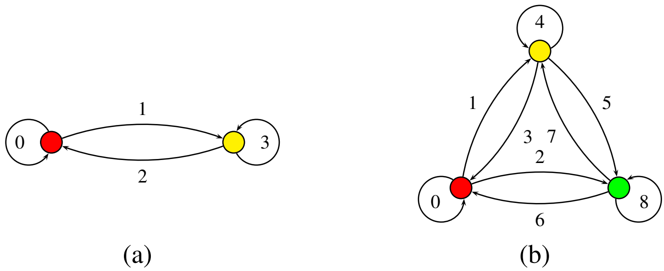


Figure 4: The complete directed graph representing the complete tile set over (a) 2 and (b) 3 colors.

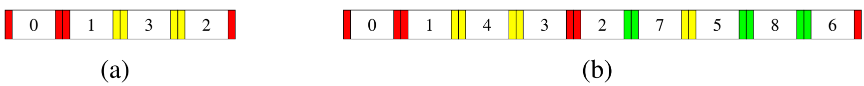


Figure 5: A tile packing of the one-dimensional complete set of tiles over (a) 2 and (b) 3 colors.

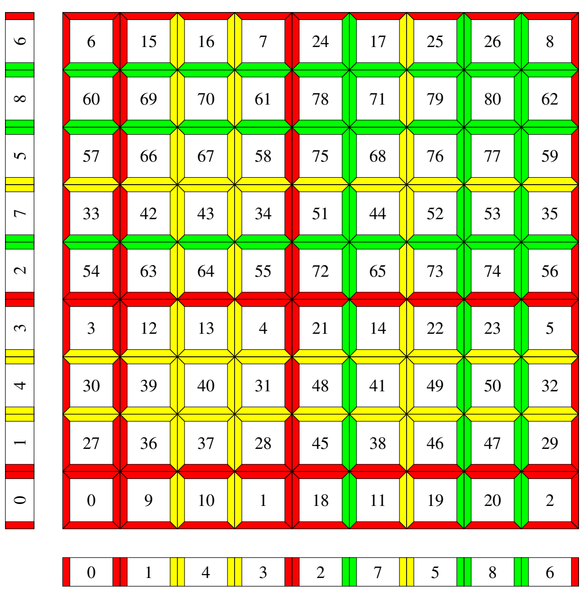


Figure 6: The outer product of two one-dimensional Eulerian tile packings over 3 colors results in an Eulerian tile packing of the complete set of edge tiles over 3 colors.

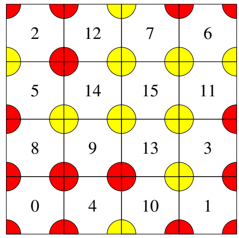


Figure 7: A tile packing of the complete set of corner tiles over 2 colors.

4 The Corner Tile Packing Problem

An edge of an edge tile constrains only a single dimension of a tiling, while a corner of a corner tile constrains the tiling in all its dimensions. This means that a solution for the two-dimensional corner tile packing problem can most likely not be obtained from a solution of the one-dimensional tile packing problem.

When we started exploring the corner tile packing problem, it resisted all attempts to solve it. Also, it was not clear whether the corner tile packing problem even had a solution. Therefore we decided to tackle the problem using combinatorial search methods. A simple *exhaustive search* or *generate-and-test* method clearly will not work: for C colors the tiles can be arranged in $C^4!$ ways, which is approximately 2.0923×10^{13} for $C = 2$ and 5.7971×10^{120} for $C = 3$. Instead we use a *backtracking* method, that places one tile at a time until a dead end is reached, at which point previous steps are retraced. The algorithm can also be used to search for solutions to the edge tile packing problem. Backtracking greatly reduces the amount of work in an exhaustive search, and is often used to solve hard combinatorial problems such as the *knights tour problem* and the *queens problem* [Bal26]. For a detailed discussion of our algorithm, we refer to [LD06b].

It turns out that for 2 colors the corner tile packing problem has 32 solutions. In contrast, the edge tile packing problem has 203520 solutions for 2 colors. This supports the claim that in some way, corner tile packings are more difficult to construct than edge tile packings. Figure 7 shows a solution to the corner tile packing problem for 2 colors.

colors	2	3	4
edge tile packing	< 1 sec	< 1 sec	23 years
recursive edge tile packing	< 1 sec	< 1 sec	140 days
corner tile packing	< 1 sec	< 1 sec	280 days
recursive corner tile packing	< 1 sec	< 1 sec	190 days

Table 1: The CPU time needed to compute the first solution to various tile packing problems.

The tile packing problem has many symmetries. New solutions can be obtained from existing ones using translation (the tile packing is toroidal), rotation, reflection, and permutation of the colors. The 32 solutions of the 2 color corner tile packing problem reduce to a single fundamental solution.

With the backtracking algorithm, we are able to compute edge and corner tile packings for 2, 3 and 4 colors. Table 1 shows the CPU time needed to compute the tile packings. For 2 colors, all edge tile packings are computed in less than a second. Constructing all corner tile packings only takes a couple of milliseconds. For 3 colors, the first solution to the edge and corner tile packing problems is obtained within a second, but computing or counting all solutions seems to be hopeless. For 4 colors, computing a corner tile packing took 280 days of CPU time, and it took roughly 23 years of CPU time to find the first solution of the edge tile packing problem. These results were obtained using a parallel version of our backtracking algorithm, running on a cluster with almost 400 2.4 GHz CPU's.

A solution for C colors can often be found faster by starting from a solution of $C-1$ colors. That way, a *recursive tile packing* is obtained. Figure 8 shows a recursive corner tile packing for 4 colors. Note that Eulerian edge tile packings are recursive. For a complete set of solutions, we refer to [LD06b].

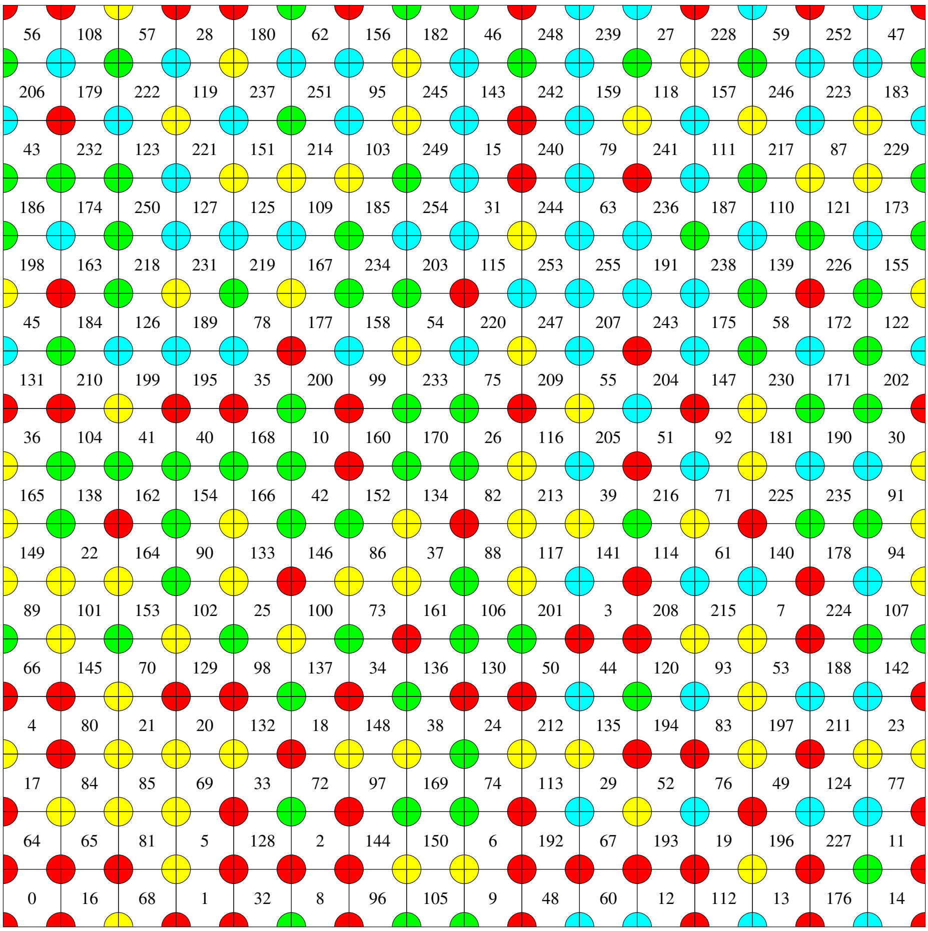
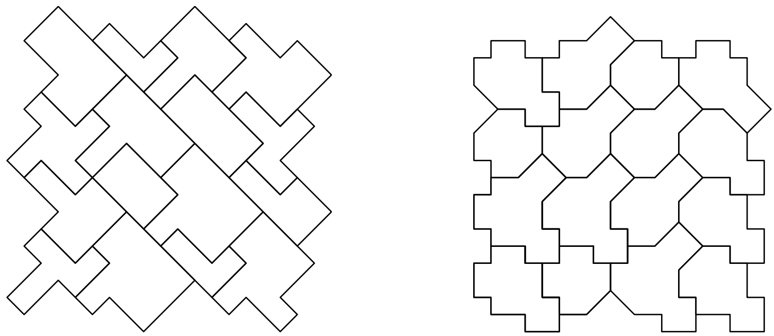


Figure 8: A recursive tile packing of the complete set of corner tiles over 4 colors. The CPU time needed to compute this solution was approximately 190 days.



(a) (b)

Figure 9: Jigsaw puzzles created from (a) edge and (b) corner tile packings of a complete tile set over 2 colors.

5 Puzzles Derived from the Tile Packing Problem

Inspired by MacMahon's work [Mac21], we have created jigsaw puzzles from the tile packings by altering the profile of the adjoining edges or corners. Figure 9 shows some examples. To create interesting puzzles, it is better to use tile packings constructed with the backtracking algorithm instead of Eulerian tile packings. We found it already hard to construct a tile packing of the complete set of corner tiles over 2 color by hand, so these puzzles should be challenging, especially puzzles based on tile packings of 3 or 4 colors. To prevent the tiles from being rotated, a picture could be printed on the puzzle.

6 Conclusion

In this paper we have introduced the tile packing problem. We have discussed how to construct an edge tile packing for an arbitrary number of colors, we have presented solutions to the corner tile packing problem, and we have shown how to create interesting jigsaw puzzles. Several open problems, such as a counting the number of solutions to the tile packing problem, and finding a constructive method for obtaining a corner tile packing, remain. We hope that this work will stimulate further interest in this puzzle amongst readers, and that the remaining open problems will eventually be solved.

Acknowledgments

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