Semantics of Definite Programs

Interpretations

1. Given the formula F: `equal(plus(X,Y),plus(Y,X))`

   - Find an interpretation I based on a finite domain and variable assignments V_1 and V_2 such that:
     a. $I \models_{V_1} F$
     b. $I \not\models_{V_2} F$

     There are many solutions, here a simple one:
     Pre-interpretation: J:
     $D = \{a, b\}$
     $plus_J : plus_J(a, a) = a, plus_J(a, b) = a, plus_J(b, b) = b, plus_J(b, a) = b$
     Interpretation I based on J: $equal_I = \{\langle a, a\rangle, \langle b, b\rangle\}$

     (a) variable assignment V_1: $\sigma = \{X = a, Y = a\}$
     $equal_I(plus_J(a, a), plus_J(a, a)) = equal_I(a, a) \in equal_I$ thus $I \models_{V_1} F$

     (b) variable assignment V_2: $\sigma = \{X = a, Y = b\}$
     $equal_I(plus_J(a, b), plus_J(b, a)) = equal_I(a, b) \notin equal_I$ thus $I \not\models_{V_2} F$

   - Find interpretations I_1 and I_2 based on an infinite domain such that:
     a. $I_1 \models \forall F$
     b. $I_2 \not\models \forall F$

     There are again many solutions, the most convenient infinite domain is the natural numbers (one could also take the Herbrand Universe which is introduced later in the course). There are solutions for (a) and (b) based on the same pre-interpretation.

     Pre-interpretation J:
     $D = N$ (the natural numbers)
     $plus_J : plus_J(X, Y)) = X + Y$

     (a) Interpretation I based on J: $equal_I = \{\langle n, n\rangle | n \in N\}$
     For arbitrary $n, m \in N : equal_I(plus_J(n, m), plus_J(m, n)) = equal_I(n+m, m+n) \in equal_I$ thus $\forall equal(plus(X, Y), plus(Y, X))$ evaluates to true and $I \models F$. (One could also take e.g. $equal_I = \{\langle n, m\rangle | n, m \in N\}$.)
     (b) Interpretation I based on J: $equal_I = \{\langle n, m\rangle | n \neq m\}$
     Consider the variable assignment $\{X = 1, Y = 2\}$: $equal_I(plus_J(1, 2)), plus_J(2, 1)) = equal(3, 3) \notin equal_I$ thus $\forall equal(plus(X, Y), plus(Y, X))$ evaluates to false and $I \not\models F$. (One could also take e.g. $equal_I = \{\emptyset\}$.)

2. Given the clausal theory T:
`even(0)` $\leftarrow$
`even(s(X))` $\leftarrow$ `odd(X)`
`odd(s(X))`$\leftarrow$ `even(X)`
`false` $\leftarrow$ `even(X),odd(X)`
Find interpretations I_1 and I_2 based on a finite domain such that:
a. $I_1 \models T$
b. $I_2 \not\models T$
Is the theory inconsistent, valid or satisfiable?

   (a) Pre-interpretation J: $D = \{e, o\}$
   $0_J = e, s_J(e) = o, s_J(o) = e$
   Interpretation I based on J:
   $even_I = \{< e >\}, odd_I = \{< o >\}$
   $I \models T$ because all clauses are true under all possible assignments. For example, the last clause: $false \leftarrow even(e), odd(e)$ reduces to $false \leftarrow false, true$ which is true, and similarly, $false \leftarrow even(o), odd(o)$ reduces also to $true$.

   (b) Pre-interpretation J: $D = \{a\}$
   $0_J = a, s_J(a) = a$
   Interpretation I based on J:
   $even_I = \{< a >\}, odd_I = \{< a >\}$
   $I \not\models T$ because `false`$\leftarrow$`even(X),odd(X)` reduces to `false`$\leftarrow$`true,true` under the only possible assignment {X=a,Y=a} which is equivalent to `false`.

The theory is satisfiable. Note, because there is a model, a proof procedure searching for inconsistency cannot succeed. Hence the SLD-proof procedure (introduced later in the course) cannot find a refutation for the program consisting of the first three lines and $\leftarrow$ `even(X),odd(X)` as goal (it will not terminate).

Substitutions and Unifications

1. Given the substitutions $\sigma = \{X/f(Z), Y/g(U)\}$ and $\tau = \{X/f(a), Y/g(W), U/W\}$.

   (a) Is σ more general than τ?
   No, θ should at least have the components $\{Z/a, U/W\}$, then $\sigma\theta$ cannot be equal to τ.

   (b) Give another substitution similar to σ which is more general than τ.
   With $\sigma' = \{X/f(a), Y/g(U)\}$ and $\theta = \{U/W\}$, $\sigma'\theta = \tau$.

   (c) Give another substitution similar to τ such that σ is more general than it.
   With $\tau' = \{X/f(a), Y/g(W), Z/a, U/W\}$ and $\theta = \{Z/a, U/W\}$, $\sigma\theta = \tau'$.

2. Apply the three unification algorithms on the following pairs of terms:

   (a) $p(X, a)$ and $p(b, Y)$

i. Robinson:

σ	A	B	D
ϵ	p(X,a)	p(b,Y)	{X, b}
$\{X/b\}$	p(b,a)	p(b,Y)	{a, Y}
$\{X/b, Y/a\}$	p(b,a)	p(b,a)	{}

ii. Paterson and Wegman:
Mark the terms as $p^1(X^2, a^3)$ and $p^4(b^5, Y^6)$
build equivalence class $1 \sim 4$
propagate $\{1, 4\} : 2 \sim 5, 3 \sim 6$
Extract substitution $\{X/b, Y/a\}$

iii. Martelli-Montanari:

{ p(X,a) = p(b,Y) }	peel
{ X=b, a=Y }	switch second eq.
{ X=b, Y=a }	solved form

(b) $p(a)$ and $p(b)$

i. Robinson:

σ	A	B	D
ϵ	p(a)	p(b)	{a, b}

No unifier (different constants).

ii. Paterson and Wegman:
Mark the terms as $p^1(a^2)$ and $p^3(b^4)$
build equivalence class $1 \sim 3$
propagate $\{1, 3\} : 2 \sim 4$
The class $\{2, 4\}$ (a, b) is not homogeneous thus failure.

iii. Martelli-Montanari:

{p(a) = p(b) }	peel
{a = b}	peel
fail	

(c) $p(X, f(g(X)))$ and $p(f(Y), f(Y))$

i. Robinson:

σ	A	B	D
ϵ	p(X,f(g(X)))	p(f(Y),f(Y))	{X, f(Y)}
$\{X/f(Y)\}$	p(f(Y),f(g(f(Y))))	p(f(Y),f(Y))	{g(f(Y)),Y}

No unifier (occur check).

ii. Paterson and Wegman:
Mark the terms as $p^1(X^2, f^3(g^4(X^2)))$ and $p^5(f^6(Y^7), f^8(Y^7))$ build equivalence class $1 \sim 5$
propagate $\{1, 5\} : 2 \sim 6, 3 \sim 8$
propagate $\{2, 6\}$: nothing
propagate $\{3, 8\} : 4 \sim 7$
propagate $\{4, 7\}$: nothing
Dependencies: $\{2, 6\} \to \{4, 7\} \to \{2, 6\}$ thus fail.

iii. Martelli-Montanari:

{p(X,f(g(X))) = p(f(Y),f(Y))) }	peel
{X = f(Y), f(g(X)) = f(Y) }	peel second
{X = f(Y), g(X) = Y }	substitute first
{X = f(Y), g(f(Y)) = Y }	switch second
{X = f(Y), Y = g(f(Y)) }	substitute second - fail (occur check)

SLD-trees

Consider the program:
p(X,Y) ← q(X,Y), r(Y).
q(X,h(Y)) ← q(X,Y).
r(g(X)) ←.
r(a) ←.
and the query ← p(X,Y).
Choose a selection rule and draw the SLD-tree. What can you conclude about the query.

(We do "selective" renaming to minimise number of variables and substitution components.)
←p(X,Y).
←q(X,Y),r(Y). {Y/h(Y1)}
←q(X,Y1),r(h(Y1)).
fail.

The query fails.
Note that the left to right computation rule (as in Prolog) leads to an infinite tree; from that tree one can only conclude that there is no refutation. Later we will see that the latter tree is "unfair".

Herbrand models, T_P operator

Let $D = \{N\}$. $J(a) = 0$, $J(f(n)) = n + 1$ and
$p_I = \{< n > \mid n \text{ is even}\}$. Let F be $p(X) \rightarrow p(f(f(X))$.
Show that $I \models F$
Construct I_H
Show that $I_H \models F$

Consider $X = n$. Interpreted under I, the formula F simplifies to $p(n) \rightarrow p(n + 2)$.
For n even, this give $true \rightarrow true$ which is true
For n odd, this gives $false \rightarrow false$ which is true
Hence F is true for all value assignments of X and $I \models F$.

Use $f^n(a)$ as abbreviation for $f(\ldots(a)\ldots)$ (n times) ($f^0(a) = a$).
$I_H = \{p(f^n(a)) | p(n) \in p_I\} = \{p(f^n(a)) | \ n \text{ is even}\}$.

Consider $X = f^n(a)$. Interpreted under I_H, the formula F simplifies to $p(f^n(a)) \rightarrow p(f^{n+2}(a))$.
For n even, this give $true \rightarrow true$ which is true
For n odd, this gives $false \rightarrow false$ which is true
Hence F is true for all value assignments of X and $I_H \models F$.

For each of the following programs (in the language underlying the program):

i. Give the Herbrand base B_P.
ii. Give all Herbrand models.
iii. Give the least Herbrand model ($T_P \uparrow \omega$).
iv. Give $T_P \downarrow \omega$ (is it a fix-point?).
v. Is T_P downward continuous?

1. p(X) ← q(X).
   q(a) ←.
   r(X) ← s(X).

   (a) $B_P = \{p(a), q(a), r(a), s(a)\}$
   (b) All Herbrand models - $\{p(a), q(a)\}, \{p(a), q(a), r(a)\}, B_P$
   (c) LHM $T_P \uparrow \omega = \{p(a), q(a)\}$
   (d) $T_P \downarrow 1 = \{p(a), q(a), r(a)\}$
       $T_P \downarrow 2 = \{p(a), q(a)\}$
       $T_P \downarrow 3 = \{p(a), q(a)\} = T_P \downarrow \omega$ (fix point)
   (e) T_P is downward continous

2. p(X) ← p(X).
   r(X) ← s(X).
   s(a) ←.

   (a) $B_P = \{p(a), r(a), s(a)\}$
   (b) All Herbrand models - $\{r(a), s(a)\}, B_P$
   (c) LHM $T_P \uparrow \omega = \{r(a), s(a)\}$
   (d) $T_P \downarrow 1 = T_P(B_P) = \{p(a), r(a), s(a)\} = B_P$ (fix point)
   (e) T_P is downward continous

3. p(f(X)) ← p(X).
   q(a) ← p(X).

   (a) $B_P = \{p(a), p(f(a)), \ldots, p(f^n(a)), \ldots\} \cup \{q(a), q(f(a)), \ldots, q(f^n(a)), \ldots\}$
   (b) Let $P^k = \{p(f^k(a)), p(f^{k+1}(a)), p(f^{k+2}(a)), \ldots\}$ and $Q = \{q(f(a)), q(f(f(a))), \ldots\}$.
       Then all Herbrand models are: $\emptyset$, $\{q(a)\} \cup Q' \cup P^k$ for any $Q' \subseteq Q$ and $k \geq 0$.
       $(B_P = \{q(a)\} \cup Q \cup P^0)$
   (c) LHM $T_P \uparrow \omega = \emptyset$
   (d) $I_0 = T_P \downarrow 0 = B_P$
       $I_1 = T_P \downarrow 1 = \{q(a), p(f(a)), p(f(f(a))), \ldots\}$
       $I_k = T_P \downarrow k = \{q(a), p(f^k(a)), p(f^{k+1}(a)), \ldots\}$
       $I_\omega = T_P \downarrow \omega = \{q(a)\}$ (not a fix point)

(e) $I_0 \supseteq I_1 \supseteq \ldots$
$glb(I_0, I_1, \ldots) = \bigcap I_0, I_1, \ldots = T_P \downarrow \omega = \{q(a)\}$ (not a fix point)
$T_P(glb(I_0, I_1, \ldots)) = \emptyset$
$glb(T_P(I_0), T_P(I_1), \ldots) = glb(I1, I2, \ldots) = \{q(a)\}$
So T_P is not downward continuous.

4. p(Y) ← p(X), q(X,Y).
p(a) ←.
q(a,b) ←.
q(c,d) ←.

   (a) $B_P = \{p(a), p(b), p(c), p(d),$
   $q(a,a), q(a,b), q(a,c), q(a,d), q(b,a), q(b,b), q(b,c), q(b,d),$
   $q(c,a), q(c,b), q(c,c), q(c,d), q(d,a), q(d,b), q(d,c), q(d,d)\}$
   (b) All Herbrand models - $\{p(a), p(b), q(a,b), q(c,d)\}, \ldots, B_P$
   (c) LHM $T_P \uparrow \omega = \{p(a), p(b), q(a,b), q(c,d)\}$
   (d) $T_P \downarrow 1 = \{p(a), p(b), p(c), p(d), q(a,b), q(c,d)\}$
   $T_P \downarrow 2 = \{p(a), p(b), p(d), q(a,b), q(c,d)\}$
   $T_P \downarrow 3 = \{p(a), p(b), q(a,b), q(c,d)\} = T_P \downarrow 4 = T_P \downarrow \omega$ (fix point)
   (e) T_P is downward continous

Soundness and Completeness of SLD

Given some program P. What can you say about
i. the inconsistency
ii. the existence of an SLD-refutation
in each of the following cases:

1. $P \cup \{\leftarrow p(f(X))\}$ is inconsistent then:
   (a) $P \cup \{\leftarrow p(X)\}$
      i. is inconsistent (1(a)ii + soundness of SLD).
      ii. By completeness of SLD there exists a refutation for $\leftarrow p(f(X))$ and by lifting lemma there also exists a refutation for $\leftarrow p(X)$.
   (b) $P \cup \{\leftarrow p(f(X))\}$
      i. is inconsistent (given).
      ii. has an SLD-refutation (completeness of SLD).
   (c) $P \cup \{\leftarrow p(f(a))\}$
      i. nothing
      ii. nothing
2. If $P \cup \{\leftarrow p(f(X))\}$ has an SLD-refutation then:
   (a) $P \cup \{\leftarrow p(X)\}$

   i. is inconsistent (2(a)ii + soundness of SLD).
   ii. has an SLD-refutation (lifting lemma).

(b) $P \cup \{\leftarrow p(f(X))\}$

   i. is inconsistent (soundness of SLD).
   ii. has an SLD-refutation (given).

(c) $P \cup \{\leftarrow p(f(a))\}$

   i. nothing
   ii. nothing

3. $P \models p(f(b))$ means that $P \cup \{\leftarrow p(f(b))\}$ is inconsistent, which means that $P \cup \{\leftarrow p(f(b))\}$ has an SLD-refutation (by completeness of SLD). Then:

   (a) $P \cup \{\leftarrow p(X)\}$

      i. is inconsistent (3(a)ii + soundness of SLD).
      ii. has an SLD-refutation (by lifting lemma).

   (b) $P \cup \{\leftarrow p(f(X))\}$

      i. is inconsistent (3(b)ii + soundness of SLD).
      ii. has an SLD-refutation (by lifting lemma).

   (c) $P \cup \{\leftarrow p(f(a))\}$

      i. nothing
      ii. nothing